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THE MATHEMATIZING MODE RE-EXAMINED

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by

YVONNE TSCHOFEN

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "The Mathematizing Mode Re-Examined," submitted by Yvonne Tschofen, in partial fulfilment of the requirements for the degree of Master of Education.

ABSTRACT

The purpose of this study was to develop a unit of work for the Mathematics 10 program using a method called the "mathematizing mode," to describe its implementation in the classroom, and on the basis of this description, to extract some general principles concerning the implementation. This method was developed as the result of work done by a group of professors and graduate students in 1967 - 1968 and is specifically described by Johnston. (Johnston, 1968)

In order to fill the need for more examples of the specific use of the mathematizing mode, a unit on factoring for the Mathematics 10 course was developed and taught to a group of seventeen students. These lessons were observed and recorded by the writer. They were also video-taped.

The aims, content, and organization of the unit are described. The unit is also described in terms of its implementation in the classroom - that is, teacher and pupil behaviour is described and discussed in detail.

An analysis of the extent of student participation in teacher-led discussions is included. Also included are the results of an attempt to classify the activities according to categories as defined by Johnston. (Johnston, 1968)

The result is a description of a unit on factoring using the "mathematizing mode;" a description and discussion of its implementation in the classroom; and on the basis of this,

a series of general principles concerning the mathematizing mode as generally employed.

No formal evaluation of the method is given, although some of the students' comments are included to give an indication of the success of the method from their point of view.

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TABLE OF CONTENTS

Page

LIST OF TABLES x

CHAPTER

I. THE PROBLEM 1

Introduction 1

Statement of the Problem 5

Need for the Study 5

Delimitations of the Study 6

Definitions 7

Summary 8

II. REVIEW OF SELECTED RELATED LITERATURE 9

Introduction 9

The Background 9

Translating Theory Into Practice 13

Beberman 13

Suchman 15

Biggs and MacLean 17

Davis 18

The Mathematizing Mode Project 23

Conclusion 26

III. THE DESIGN OF THE STUDY 30

Introduction 30

The Problem 30

CHAPTER		Page
	The Instructional Setting	31
	The Students	31
	The Physical Setting	31
	The Teacher	32
	The Content	32
	The Procedure	34
IV.	THE MATHEMATIZING MODE IN PRACTICE	38
	Introduction	38
	The Objectives of the Unit	39
	The Objectives of the Exercises	41
	Exercise #1	42
	Exercise #2	42
	Exercise #3	43
	Exercise #4	43
	Exercise #5	44
	Exercise #6	45
	A Description of the Unit on a Day to Day Basis .	46
	Day 1	46
	Day 2	48
	Day 3	49
	Day 4	50
	Day 5	51
	Day 6	51
	Day 7	52

CHAPTER Page

Day 8 52

Day 9 52

Summary 53

V. AN ANALYSIS OF THE MATHEMATIZING MODE IN PRACTICE . 56

Introduction 56

A Discussion of the Unit on a Day to Day Basis . . 56

Day 1 57

Day 2 64

Day 3 66

Day 4 70

Day 5 73

Day 6 76

Day 7 78

Day 8 82

Day 9 84

The Behaviour of the Students in Terms of Participation in Discussion Sessions 85

Background 85

The Results 86

Discussion of the Results 89

A Classification of the Classroom Experiences . . 90

The Mathematizing Mode as Described by Johnston 90

A Classification of the Experiences 92

An Application of the Classification 95

Discussion of the Results 96

CHAPTER Page

Summary and Conclusion 98

VI. IMPLICATIONS, SUMMARY, AND CONCLUSIONS 99

Introduction 99

A Discussion of the Implications of Chapter V . . 99

Expectations of the Teacher 100

General Characteristics of the Lessons and Exercises 103

Implications of the Analysis of Student Participation in Discussion Sessions 113

A Summary and Discussion of the Results of the Classification 118

Limitations of the Study 121

Suggestions for Further Research 124

Summary 125

Conclusion 126

BIBLIOGRAPHY 129

APPENDIX A

Teacher and Pupil Behaviour 134

LIST OF TABLES

TABLE		Page
I.	Summary of Activities of Day 1 to Day 4	54
II.	Summary of Activities of Day 5 to Day 9	55
III.	Exercise #1	58
IV.	Exercise #2	68
V.	Exercise #3	71
VI.	Exercise #4	74
VII.	Exercise #5	81
VIII.	Exercise #6	83
IX.	Student Rankings	87
X.	A Classification of the Unit	97
XI.	Summary of Classification	98

CHAPTER I

THE PROBLEM

I. Introduction

There has been a great deal of criticism of traditional teaching methods, especially in the field of mathematics. Glaymann sums up these criticisms very succinctly:

> . . . il est primordial que la mathématique, science vivante et dynamique par excellence, soit enseignée avec efficacité. Et cependant, dans la plupart des pays, elle est encore enseignée telle une langue morte, coupée de ses applications les plus fécondes et trouvant en elle-même sa fin. En outre, les élèves sont enfermés dans des dogmes austères et souvent stériles; l'enseignement traditionnel ne tient nullement compte de la sensibilité de l'enfant, ni de sa forme de pensée et encore moins de sa puissance créatrice. Il est privé de toute liberté, il doit souvent, de manière purement passive, apprendre avant même de comprendre. (Glaymann, 1968, p. 118)

Is mathematics still being taught as a dead language, a closed system, in a way that a child must "learn" before he ever understands? New teaching methods which subordinate retention to thinking have suddenly become imperative and fashionable. "How" to teach mathematics has become as important as "what" to teach in mathematics, and of more importance, "how" the student learns has become as important as "what" he learns. Mathematics as a way of thinking has supposedly replaced mathematics as a body of knowledge to be transmitted. This shift, at least in theory, from content objectives to process objectives has found champions such as Schwab, and Parker and Rubin:

> Content is a rhetoric of conclusions to be transferred to the student. (Schwab, 1968, p. 131)

> Process, in contrast, refers to all the random, or ordered, operations which can be associated with knowledge and with human activities. There are a variety of processes through which knowledge is created. There are also processes for utilizing knowledge and for communicating it. Processes are involved in arriving at decisions, in evaluating consequences, and in accommodating insights. The scientist engages in what is perhaps the crucial process of his labor when he fabricates questions for which answers must be found . . .
>
> Where primary emphasis is upon content, the learner ordinarily functions in the passive mode. He conditions himself to submit to authority. He accepts the proffered gospel, and he neither selects his conclusions nor assesses their validity. He does not wear a tailor-made mind, but a ready-made one, cut in the fasion of the day. Even here he employs a number of processes -- directed toward the sponging-up of bookishness and to its consequent exhibition in the preferred manner.
>
> Where the stress is upon process, the assimilation of knowledge is not derogated, but greater importance is attached to the methods of its acquisition and to its subsequent utilization. (Parker and Rubin, 1968, p. 131)

And so, in theory at least, educators - teachers, curriculum makers, authors of texts, et cetera, seem to attach more importance than ever to the methods of acquisition of knowledge. The effects of this current thought, however, have been slow to reach the classroom, and it is meaningless to talk of change in education, unless it affects very directly, the students in the classroom. Very little help has been given to the teacher in translating the theory of a process-oriented teaching method into practice. Instead, the teachers have been bombarded with more theory - theory concerning team teaching, laboratories, individualized instruction, and so on. These methods all purport, at least to some extent, to stress the importance of processes.

In 1967-68, a group of professors and graduate students in the Department of Secondary Education at the University of Alberta, recognized the need for a well-defined teaching method - one which emphasized mathematics as an activity rather than as the contemplation of a "fait accompli," and for a specific example of its implementation which could serve as a model for teachers interested in the method. As a result, a specific type of discovery teaching called the "mathematizing mode" was developed and described; a unit of mathematics for grade eleven using this method was created; and teachers were trained to implement it in the classroom. The method was then compared to the traditional or expository method of teaching. Although the word "processes" was hardly, if ever, mentioned, the emphasis of the method was definitely on having the students "invent," or "create," or "discover" the mathematics required in a given situation. The description of the classroom activities was given in these terms, but the evaluation was more in terms of the products of the method.

The writer felt that there still existed a need for more examples of the specific use of the mathematizing mode in the classroom. Information concerning the aims, content, exercises, teacher behaviour, and pupil behaviour in a specific unit of mathematics could be of great use to anyone wishing to use the method. By being thoroughly acquainted with many specific examples of the use of the mathematizing mode, teachers could become familiar with its philosophy, goals, and techniques.

Certain aspects of the implementation of the method were of interest to the writer, in particular, the actual amount of participation of the students in teacher-led discussions. What percentage of the time would the students be involved in teacher-led discussions, and what percentage of the time in working on their own? Would the discussions be dominated by certain students? If so, would these students be the high achievers? The most intelligent students? The boys? The girls?

Johnston (Johnston, 1968) described the activities of a unit using the mathematizing mode in terms of four stages. The writer wondered about the general suitability of this description. If these stages actually existed in practice in other units using the mathematizing mode, how much time was spent on each stage?

As the result of a close examination and analysis of an implementation of the mathematizing mode in the classroom, could generalizations concerning the method be made? The writer felt that this was possible and that it would be useful to compare such generalizations to those made by Johnston. (Johnston, 1968)

Rather than observing many teachers who were perhaps not so much at ease with the mathematizing mode, the writer believed it to be better to observe one teacher who was well informed of the theory, and experienced in the implementation of the method. Thus, the implementation of the mathematizing mode could be observed and documented under fairly ideal conditions.

II. Statement of the Problem

The purpose of this study was to create a unit of mathematics using the mathematizing mode, to describe its implementation in the classroom, to analyze specific aspects of these classroom experiences, and to produce generalizations concerning the mathematizing mode.

To accomplish this, a unit of study, using the mathematizing mode, for a grade ten Mathematics 10 class was prepared and presented to a group of students. The unit and its implementation is described in terms of its aims, content, exercises, teacher behaviour, and pupil behaviour. Selected aspects of the unit are discussed and analyzed, and generalizations based on these discussions and analyses are made and compared to those made by Johnston. (Johnston, 1968) These generalizations deal with implied teacher expectations, organization and structure of the unit, participation of the students in teacher-led discussions, and the description of the mathematizing mode in terms of stages as described by Johnston. (Johnston, 1968)

III. Need for the Study

What sort of educational research most benefits the classroom teacher? Bruner answers:

> Develop the best pedagogy you can. See how well you can do, then analyze the nature of what you did that worked. We do not yet have enough good principles at this point to design an adequate experiment in which this group gets this treatment and that group another treatment. The experiments of this type have been grossly disappointing. The best things that you can do at any given point, I would urge, is to design a pedagogical treatment that works extremely well, and then work your way back. Later on design hypotheses to determine what you did. But for the moment, can we not declare a

> moratorium on little experiments that produce miniscule effects? . . . With a mixture of psychology, common sense, and luck you may produce an effect on learning that is worth studying. Then purify and experiment. But first invent and observe. That seems to me to be the pragmatic strategy. It is not in the grand experimental tradition of physics. But is experimental pedagogy in the grand tradition of physics at this point in history? It may very well be that it is more like economics, a mixture of models and pragmatics. . . . The formal experiments can wait until we have shown that some treatment is worth the trouble. (Bruner, 1961, p. 113)

Although Johnston (Johnston, 1968) has already described the theoretical framework of the mathematizing mode and its implementation in the classroom, there still existed a great need for many more specific examples of translating this theory into practice. Part of this study is devoted to describing how the mathematizing mode was again implemented in the classroom. The content used was different from that used in any of the previous studies. There existed also a need to re-examine and analyze the mathematizing mode to further identify crucial teacher and pupil behavour. Sigurdson says:

> Although it has been common practice to develop a theory of learning or concept development, and then to suggest implications from the theory regarding teaching practices, it is possible to proceed in the reverse direction, namely to identify good teaching practices and, on the basis of these, to generate theory - not only a theory of instruction but also of the psychology of learning and motivation. (Sigurdson and Johnston, 1970, p. 121)

The intent was to add to, to modify, or to reject the existing theory concerning the mathematizing mode.

IV. Delimitations of the Study

The teacher presented a unit on factoring to a Mathematics 10 class in nine eighty minute lessons. The group of students consisted of seventeen volunteers from an existing Mathematics 10

class. In order to examine the mathematizing mode in a fairly ideal situation, "chronic skippers" and those repeating the course were excluded.

The lessons were video-taped, the focus of these tapings being mainly on the teacher. The descriptions of the lessons deal primarily with what went on during the teacher-led discussions. Very little of the pupil interaction during the work periods is given.

No formal attempt is made to evaluate the method. Therefore no test marks are included.

V. Definitions

Mathematizing mode. This term refers to a specific type of discovery learning, one which was described by Johnston. (Johnston, 1968) It is the method which will be used in this study.

Discussion session. This is simply a teacher-led discussion. All of the pupils are expected to pay attention and to focus on the same work.

Personal-inquiry session. This refers to a period of time in which the students work on their own, or with other pupils, or on a fairly private basis with the teacher. The students are not expected to focus on exactly the same work.

VI. Summary

The purpose of this study was to create a unit of mathematics using a process-oriented method called the mathematizing mode; to examine the implementation of this unit in the classroom; to discuss and analyze selected aspects of these classroom experiences; to extract some generalizations concerning the method; and where possible, to compare these generalizations to those made by Johnston. (Johnston, 1968)

CHAPTER II

REVIEW OF SELECTED RELATED LITERATURE

I. Introduction

In this chapter, some general views concerning the discovery learning method will be presented. The writer will then consider some of the efforts which have been made to translate existing theory into practice. This will include the attempts of Beberman, Suchman, Biggs and MacLean, and Davis and a description of the Mathematizing Mode project begun at the University of Alberta in 1967-68.

II. The Background

> The traditional arithmetic curriculum could be attacked as, among other things, a menace to mental health; an abstract ritual devoid of meaning, to be learned precisely with no allowance for error or originality; it has been denounced by children and adults alike since the time of . . . To this mainly negative ritual there was added the further insult of some exceedingly middle class (and quite wrong) discussions about buying annuities, investing money, and managing checking accounts. (Davis, 1967b, p. 65)

The trend towards learning by discovery may in part be the result of a rebellion against the situation described above.

What is discovery learning? Bittinger defined it as "any learning situation in which the learner completes a learning task without extensive help from the teacher." (Bittinger, 1968, p. 140) He claims that it may entail any or all of the following methods: (1) the inductive method, (2) the nonverbal awareness method, (3) the incidental learning method, (4) the deductive method, (5) the variation method.

Bolding seems to relate the discovery method quite specifically to the inductive method in which students are "discovering a property as the result of well-planned exercises." (Bolding, 1964, p. 105) He illustrates with an example in which the student is led to discover the distributive property:

$2 \times 4 + 2 \times 6 = 20 = 2 \times 10$

$7 \times 8 + 7 \times 10 = 126 = 7 \times$ _____

$3\frac{1}{2} \times 4 + 3\frac{1}{2} \times 6 = 3\frac{1}{2} \times$ _____ $= 3\frac{1}{2} \times (4 +$ _____$)$

$9/8 \times 2 + 9/8 \times 1 = 9/8$ (_____ + _____)

$\square \times 0 + \square \times \Delta =$ _____

$x \cdot y +$ _____ $= x (y + z)$

(Bolding, 1964, p. 105)

To the inductive method, Hendrix adds the dimension of non-verbal awareness. She takes the extreme view that verbalizing a generalization can actually diminish the power to transfer to new situations. (Hendrix, 1947, 1961) Other researchers take the more moderate view that verbalization of a general principle must be delayed until the teacher is certain that the generalization has been made.

Dewey and the Progressive Educationists were largely responsible for associating the act of discovery to the inductive and incidental methods of teaching in which students learned basic principles as by-products of other learning projects in which they were involved.

The deductive method is not usually associated with discovery learning, and the method of variation can be associated with all types of teaching. It is, of course, very possible to have students

discover or deduce a proof simply by manipulating symbols. The method of variation can be used to discover solutions, to verify conjectures and to qualify them.

Bruner eloquently argues that

> . . . if man's intellectual excellence is the most his own among his perfections, it is also the case that the most uniquely personal of all that he knows is that which he has discovered for himself. (Bruner, 1961, p. 22)

He believes that the benefits which accrue from the experience of learning by discovery are 1) an increase in intellectual potency, 2) a shift from extrinsic to intrinsic rewards, 3) learning the heuristics of discovery, 4) an aid to memory processing. Bruner admits that he is only hypothesizing when he states that learning by discovery

> . . . has precisely the effect upon the learner of leading him to be a constructionist, to organize what he is encountering in a manner not only designed to discover regularity and relatedness, but also to avoid the kind of information drift that fails to keep account of the uses to which information might have to be put. . . . (Bruner, 1961, p. 26)

and that "Practice in discovering for oneself teaches one to acquire information in a way that makes that information more readily viable in problem solving." (Bruner, 1961, p. 26)

On the other hand, Ausubel feels that learning by discovery has become a fad and religion and that its proponents "have elevated it into a panacea making exaggerated claims for its uses and efficacy that go far beyond the evidence as well as far beyond all reason." (Ausubel, 1961, p. 18) He points out that most of what anyone knows has been discovered by others and communicated to him in a meaningful fashion, and that anything

with meaning must necessarily be personal and unique. Ausubel does believe, however, that discovery learning is suitable for problem solving, but does not feel that problem solving should be the main aim in education. He also feels that the discovery method should be used for small children and for adolescents and adults who are in the early stages of a new discipline.

Discovery learning as has been shown means a great number of things. Davis (Davis, 1967c, pp. 59-63) elaborates on the different interpretations of the purpose or essence of discovery learning. Some of the questions he asks are:

1. Is discovery learning simply a teaching strategy of having the children do something first, and then discussing it together with them after?

2. Does its essence lie in not verbalizing, or not verbalizing immediately? Is this based on the grounds that the student will contaminate his originally clear idea by inaccurate verbalization? Or, do we believe that "the student needs much practice in dredging ideas out of his intuition and clothing them in the attire of increasingly more formal explicit language?" (Davis, 1967c, p. 62)

3. Is it a teaching method that gives the teacher more and better feedback from the students?

4. Does it allow the child to realize that mathematics can be discovered?

5. "Do we value discovery because it is an attempt to sustain a process approach in a school setting where nearly every process seems to gravitate quickly into merely the rote memorization of fact?" (Davis, 1967c, p. 61)

6. Does it help keep alive the child's creativity and curiosity?

7. Does it limit "our ability to make our children grow up in our own image, and [thereby] release them to see the world with their own eyes?" (Davis, 1967c, p. 62)

8. Because a higher percentage of incorrect statements will be made, does it train the students to listen more carefully and critically to each other?

9. Is it simply a good theatrical device, a method of maintaining a certain amount of tension?

We must translate the theory of discovery learning many times into practice to determine its true essence.

III. Translating Theory Into Practice

The intent here is to discuss briefly some of the educational projects which have attempted to implement discovery techniques. Most of these programs are not attempting to determine, at least not to any great degree, how much of their success is due to actual discovery method techniques, and how much is due to well planned programs in terms of sequence, reinforcement, good teachers, etc. The following discussions will include very little on evaluation.

Beberman

As early as 1952, Max Beberman and other prominent mathematicians and educators from the University of Illinois came together to develop a mathematics curriculum and to train high school teachers in its use. They concentrated on preparing programs

for gifted grade eight and nine classes and high school classes, programs which would promote understanding. In 1958 Berberman attempted to bring to light, in *An Emerging Program of Secondary School Mathematics*, some of the major principles which guided their work. In his words, a successful program was based on this belief:

> We believe that a student will come to understand mathematics when his textbook and teacher use unambiguous language and when he is enabled to discover generalizations by himself. (Berberman, 1958, p. 4)

As an example of the precision in language which he is attempting to develop, Beberman discusses the importance of making the students aware of the distinction between things and their names - in particular between numbers and numerals.* In his program the students are carefully taught to differentiate between what are simply naming systems and what are actual entities. (Beberman, 1958, pp. 5-23) He insists, however, that the student be aware of a concept before a name is assigned to it. (Beberman, 1958, p. 33)

With reference to discovery, Beberman states that "the student will come to understand mathematics if he plays an active part in developing mathematical ideas and procedures." (Beberman, 1958, p. 24) Once a discovery has been made, Beberman does not believe that it is necessary for the student to verbalize his discovery immediately. The teacher can determine if indeed the student has made the discovery by questioning him. As a matter of

---

*Trivett makes the same point:

> To write or say 2 + 3 = 5 is a convention, a convenience. . . . The meanings behind the symbols, however are not convention. They are based on fact over which man has no control. He can only be ignorant of the fact. (Trivett, 1970, p. 9)

fact, delaying the verbalization of discoveries is an important technique of this program. It assures that:

1. all the students will have greater opportunity to make the discovery;

2. the student is not forced to make a statement when he may not have the linguistic capacity to do so;

3. a signal is not given by the teacher that the search is over; (A concluding or summarizing statement often does this.)

4. more time is given to the student to ascertain that he is truly aware of the generalization - he may be regarding the generalization as merely another isolated example. (Beberman, 1958, pp. 26-27)

Whenever possible, routine drill exercises should be tied in with activities of greater intrinsic interest. Beberman gives as an example that drill in factoring should be provided in exercises on solving quadratic equations. (Beberman, 1958, p. 31)

Throughout his book, Beberman gives examples of actual teacher-pupil experiences to illustrate the principles which he sets forth.

Suchman

J. Richard Suchman, also of the University of Illinois has developed an Inquiry Training Program to help children "comprehend the logic of inquiry and to give them practice and guidance in searching for systematic relationships in physical phenomena." (Suchman, 1964, p. 13) The students were placed in situations in which they could learn only through self-initiated and self-

implemented projects in which they formulated their own hypotheses and tested them. Suchman believes that inquiry can be divided into four types of action: searching, data processing, discovery, and verification. The data processing is of four types: analysis, comparison, isolation, and repetition. As a result of the processing, the data is assimilated, that is, interpreted in terms of pre-existing concepts, and/or the concepts are accommodated, that is modified to correspond to the data. Suchman believes that a student can be taught to shift back and forth from assimilation to accommodation efficiently. His rationale for the program, which resembles Bruner's claims for discovery learning is:

> (a) Learning through inquiry transcends learning which is directed wholly by the teacher or the textbook: the autonomous inquirer assimilates his experiences more independently. He is freer to pursue knowledge and understanding in accordance with his cognitive needs and his individual level and rate of assimilation.
>
> (b) Inquiry is highly motivated because children enjoy autonomous activity particularly when it produces conceptual growth.
>
> (c) Concepts that result from inquiry are likely to have greater significance to the child because they have come from his own acts of searching and data processing. They are not just abstractions that have been structured by the words of other people. They are formed by the learner himself from the data he has collected and processed himself; and for that reason should be more meaningful to him, and hence, more stable and functional. (Suchman, 1964, p. 4)

It was found, on a small scale at least, that after certain kinds of inquiry training, the students were able to ask three times as many questions as those who were not trained. Suchman found also, that learning through inquiry does not detract from (and in some cases actually enhances) the learning of physical principles, and

that there is some evidence that it is superior to expository teaching in terms of conceptual growth as measured by the P.C.E.* (Suchman, 1964, p. 55)

Biggs and MacLean

Biggs and MacLean, in their efforts to translate educational theory into practice have produced the text *Freedom to Learn*. It contains a wealth of excellent projects suitable for younger and older children in all fields of mathematics - geometry, statistics, set theory, measurement, etc., as well as an outline of projects for teacher or teacher-in-training workshops. An interesting feature of this book is that almost all of these suggested projects are given in the form of how students reacted to a particular problem. That is, these suggestions are in the form of real and tried classroom situations. The following is a typical example:

> Some eight-year-olds were asked to measure the width of the room. "It's six and a half bodies," they replied. "Whose body?" they were asked. Three friends stood up. Paul and Blair were of the same height, but Roger was at least a head taller than the other two. (They had laid themselves head-to-foot across the room.) The teacher asked them if there would be any difference if they had used Roger only. "Yes," was the reply. "Roger would be very tired." The teacher pressed them further: "But would you get six and a half 'Rogers'?" she asked. All but one insisted that the answer would still be 6½ bodies because they were measuring the same distance. Ann, however, said that there would be fewer "Rogers" because he was taller than his friends. Here was a good starting point for the need first for an equal unit and then for standard units. (Biggs and MacLean, 1969, p. 124)

---

*P.C.E. - Predict-Control-Explain Test. A research instrument devised by Suchman and Fejfon.

Biggs and MacLean outline the aims of education as being:

> 1. to free students, however young or old, to think for themselves.
>
> 2. to provide opportunities for them to discover the order, patterns and relations which are the very essence of mathematics, not only in the man-made world, but in the natural world as well.
>
> 3. to train students in the necessary skills. (Biggs and MacLean, 1969, p. 3)

It is suggested that these aims can be achieved by producing a more child-centered, activity-oriented program. The suggestions for such a program are obviously based on the Piagetian concept of development of the child.

While presenting very specific examples and techniques to be used, Biggs and MacLean emphasize that it is as much an attitude towards learning rather than a specific technique which they are trying to develop in the teacher. They insist that

> Teachers cannot be expected to encourage children to display initiative if they themselves are not allowed freedom to use their own initiative and plan programs that truly meet the needs of the children in their classes. (Biggs and MacLean, 1969, p. 66)

Teachers too must experience the excitement of exploration and discovery if they are to gain an insight into this process of learning.

Davis

Robert Davis with the Madison Project has attempted to produce a curriculum of worthwhile classroom experiences in mathematics. The required attributes of these learning experiences are:

1. The lesson must relate directly to a list of fundamental mathematical concepts. (Such a list was compiled by a group of mathematicians.)

2. The student must have an active role. The teacher almost never lectures. A discovery approach in a process-oriented lesson helps assure that the pupil is not passive.

3. Concepts must be learned in context. Everything in mathematics, including concepts, facts, techniques, etc. can arise out of the act of tackling problems, that is, out of inquiry. Davis believes that this is much better than if the topic seems to be chosen at the whim of the teacher or department. Each mathematical concept or technique should appear as a reasonable answer to a reasonable question.

4. If possible, an interesting pattern should exist. There should be the opportunity for the student to make a discovery.

5. The children must have had enough previous experiences so as to be able to learn the new material. This involves a type of "built-in readiness." All required concepts must be broken down into their "atomic" constituents and these must be carefully worked into the "leading-up" experiences.

6. The experiences must be appropriate to the ages of the children, to their background, their previous experiences in mathematics.

7. The sequence of experiences must add to something worthwhile,

that is, the child must be progressing towards mathematical maturity. (Davis, 1965, pp. 4-5)*

Kieren states that

> These experiences were, among other things, designed to provide a foundation for later work in mathematics, emphasize mathematics as a process, provide greater use of physical materials, and develop an atmosphere receptive to student initiative. Hopefully, these experiences would then provide understanding of mathematical concepts and unify arithmetical concepts while arousing interest and relating mathematics to science. (Kieren, 1969, p. 515)

These learning experiences reflect the goals which Davis has set out.

The objectives of the Madison Project are:

1. "Cognitive" or "Mathematical" objectives:

   (i) the ability to discover pattern in abstract situations;

   (ii) the ability (or propensity) to use independent creative explorations to extend "open-ended" mathematical situations;

   (iii) the possession of a suitable set of mental symbols that serve to picture mathematical situations in a pseudo-geometrical, pseudo-isomorphic fashion . . . ;

   (iv) a good understanding of basic mathematical concepts (such as variable, function, isomorphism, linearity, etc.) and of their inter-relations;

   (v) reasonable mastery of important techniques;

   (vi) knowledge of mathematical facts.

2. More general objectives:

   (i) a belief that mathematics is discoverable;

   (ii) a realistic assessment of one's own ability to discover mathematics:

   (iii) an "emotional" recognition (or "acceptance") of the open-endedness of mathematics;

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*Also from: (Davis, 1964, pp. 148-150)

(iv) honest personal self-critical ability;

(v) a personal commitment to the value of abstract rational analysis;

(vi) recognition of the valuable role of "education intuition";

(viii) a feeling that mathematics is "fun", "exciting" or challenging" or "rewarding" or "worthwhile." (Davis, 1964, pp. 158-59)

Davis' philosophy revolves around three important basic notions. The first one, in his words: "We learn by successive approximations, and there is no final or absolutely perfect ultimate version in any of our minds." (Davis, 1964, p. 153) And again he states

> . . . an orientation based upon the notion of gradual modification of the individual's internal cognitive structure appears to us as highly appropriate for studying the learning of mathematics. (Davis, 1964, p. 152)

One has only to attempt some elementary computer programming to realize that it really is "human to err" and that we never get things perfect from the start. Therefore, when teachers try to present "a perfect picture" from the start, they are ignoring the fact that we learn by assimilation and accommodation, that we must all continually modify, reject, or extend our concepts.

A second important notion in Davis' philosophy is that reinforcement should be derived, first of all, from the internal structure of the mathematics itself, and secondly, from the approval of the peer group.

And finally, Davis believes that

> To use a language for significant communication, one should use the language transparently, and not allow it to become a matter of focal attention. (Davis, 1967a, p. 14)

It is generally felt that pupils should use words and then gradually explicate the meaning for themselves from continual use. This leads into the controversy of who should verbalize the required concepts and when this should be done. Some insist that the child should never verbalize - it is like not spoiling a piece of music by discussing it too much; others insist that the child must verbalize his discovery, but only late in the learning experience; still others feel that the teacher alone is the one who can "say it correctly," and therefore he should do the verbalizing. (Davis, 1967b, pp. 54-55)

Davis seems to favour the eventual verbalization by the students - in their language.

The Madison Project is concerned, not only with producing a curriculum, but also with introducing it into the schools. To do this, the lessons are taught to groups of students, criticized, modified, until final, suitable, "polished" versions exist and are taught and recorded on video or audio tape. These tapes form the basis for teacher training. Against the argument that such fully-worked out lessons erode the role of the teacher, Davis says

> Examination of the . . . lessons should make it clear that the teacher's role is not less important in this kind of teaching, nor is the lesson format inflexible. The lessons are designed to be student-centered and to vary in response to student needs and student initiative. "Heeding the student" is not the same thing as making unnecessary errors. (Davis, 1965, p. 37)

Davis feels then, that teachers need help in planning lessons. Too often they are kept so busy that they have little time to do anything but repeat the text to the children. It takes a great deal of time and effort on the part of highly competent people to polish off

an effective learning experience which has the possibilities of being flexible.

The Madison Project then, is a large undertaking designed to produce new curricula, to introduce it into the schools, and to revise it where necessary. Only limited testing has been done to see whether or not these goals have been achieved.*

The Mathematizing Mode Project

The research most pertinent to this study was conducted in 1967-68 by five investigators from the Department of Secondary Education at the University of Alberta. The project was designed mainly to compare discovery teaching to expository teaching on several dimensions. A specific type of discovery learning was considered, one which was labelled the "mathematizing mode." Before discussing this research, the writer wishes to consider the meanings attached to the terms "mathematizing mode" and "mathematizing."

The term "mathematizing mode" was coined by Johnston (Johnston, 1968) in order to describe precisely one particular type of instruction. It is used in this study in the same capacity. The word "mathematizing" according to Johnston "implies activity on the part of the learner; it implies application of mathematics already discovered; it implies acting as a mathematician." (Johnston, 1968, p. 38)

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*Robert Cleary conducted a study in 1965 and found that the children using the Madison Project materials did better on a test which measured acquisition of skills and facts than did the control groups. (Davis, 1965, pp. 95-96)

Mathematizing is used in the sense of reducing reality to mathematical terms which are fundamental, flexible, and have more general applicability. It implies the use of the fundamental activities of: "observation, description, idealization, local logical analysis, axiomatization, application." (Steiner, 1968, p. 181) Freudenthal also speaks of mathematizing as a process:

> In its first principles mathematics means mathematizing reality, and for most of its users this is the final aspect of mathematics, too. For a few ones this activity extends to mathematizing mathematics itself. The result can be a paper, a treatise, a textbook. A systematic textbook is a thing of beauty, a joy for its author, . . .
>
> Systematization is a great virtue of mathematics, and if possible, the student has to learn this virtue, too. But then I mean the activity of systematizing not its result. Its result is a system, a beautiful closed system, closed, with no entrance and no exit. In its highest perfection it can even be handled by a machine. But for what can be performed by machines, we need no humans. What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics. (Freudenthal, 1968, p. 7)

The mathematizing mode then is a teaching method which among other things emphasizes the open-endedness of mathematics, one which is as concerned with the activity of learning as with the result, one which operates on the belief that students can solve problems without first being given a rule. Fletcher says:

> In fact, each time we explain a rule and then set exercises on applying it we teach the child more than the rule in question, we teach him also that without the rules first being explained he cannot get started. In fact we engender the very attitude of mind which later will prevent him from acquiring the independence which is essential for creative work. (Fletcher, 1968, p. 167)

The mathematizing mode was described by Johnston in terms

of four stages. (Johnston, 1968)* Johnston then developed instructional materials for students in Mathematics 20 on (1) the linear function, (2) the quadratic function. These materials were designed to implement his description of the method. The units were taught to a group of students. Johnston described these lessons as activities in which he identified various stages which he had previously defined and described.

Seven teachers from the Edmonton Public School System then taught these two units for approximately seven weeks each, to two different groups of students - one group using the expository method, the other group using the mathematizing mode. These teachers had all attended in-service training sessions designed to acquaint them with both methods. Attempts were then made to compare the two methods.

Tobert (Tobert, 1969) measured student achievement before, immediately after, and a period of time after the two treatments. He found "that the Expository group out-performed the Mathematizing group on achievement tests." (Tobert, 1969, pp. 52-53)

Naciuk (Naciuk, 1968) found that the behaviour of the teachers in a Mathematizing class was definitely different as compared to their behaviour in an Expository class.

Taylor-Pearce found that for "fluency, originality, and 'total response', the treatment effects of the expository method were superior to the treatment effect of the mathematizing method." (Taylor-Pearce, 1969, p. 106)

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*A detailed description of these four stages will be given in Chapter V, and the generalizations made by Johnston concerning the method will be reviewed in Chapter VI.

Vance (Vance, 1969) found that 38% of the Mathematizing group favoured continuing use of the Mathematizing Method, 48% favoured use of the regular method, and 14% did not care either way. There seemed to be no relation between a student's like or dislike for the method and his score on the past tests.

In spite of this poor showing for the Mathematizing Mode, Naciuk found that four out of seven teachers used significantly different methods after the experiment from those they used before the in-service treatment. (Naciuk, 1968, pp. 121-22) It would seem that these teachers found the method worthwhile enough to continue using it or an adaptation of it. Tobert suggests:

> The Mathematizing Mode was, in spite of a "warm-up" period, essentially unfamiliar to the students; that is, it was the method most different from their previous experience.
>
> Teacher unfamiliarity with the Mathematizing Mode may have been a factor. While it is known that teacher behaviours were different in a Mathematizing class compared to an Expository class, it is not known whether or not teachers performed as effectively as possible with the Mathematizing Mode. There seems little doubt that teachers were quite familiar with the Expository Mode. (Tobert, 1969, p. 56)

It would seem that the method warrants further investigation under more favourable conditions.

IV. Conclusion

Two very different aspects of learning by discovery are reflected in the literature. One has to do with finding out what the learner does when he is learning by discovery, and the other has to do with what the end product is in terms of what the student has finally learned to do.

A great deal of research has been devoted to the latter aim. Craig (Craig, 1956) concluded that guidance had a positive effect on learning by discovery in terms of getting more correct responses and greater retention. Kittel (Kittel, 1957) found that the group given a verbal as well as a written statement of the principle involved was superior, in terms of transfer, to a group given the answer as well as the principle. Gagné and Brown (Gagné and Brown, 1961) found that the guided discovery group was the most successful in completing the task of deriving the formula for the sum of n terms in a number series. Hendrix (Hendrix, 1946; 1947; 1961) using the same number task, concluded that the highest transfer effects were achieved by subjects in the groups which did not verbalize the rule which they had discovered. And, although Kersh complains that "Seldom do we find a research report of a study of discovery processes which measures much more than the learner's ability to retain what was learned" (Kersh, 1964), he too, restricts himself to discussing learning by discovery in terms of retention, transfer, and interest. (Kersh, 1958; 1962; 1964; 1965) Worthen (Worthen, 1967) in a very interesting classroom experiment in which both time sample and learning task were representative of typical school behaviour, concluded that in terms of ability to retain concepts and to transfer heuristics of problem solving, the discovery method was superior. In these studies cited then, the emphasis is on learning by discovery in order to improve the student's retention of concepts learned, or his ability to transfer generalizations. Beberman (Beberman, 1958) looks upon learning by discovery as a better way of having students learn mathematics which already

exists, and so too does Suchman when he states "Concepts that result from inquiry are likely to have greater significance to the child . . . and hence are more stable and functional." (Suchman, 1964, p. 4)

It is interesting to contrast these views to those expressed at a meeting held in Utrect in 1967 on the teaching of mathematics. Pollak complains that, as a result of our teaching methods, "Mathematics has become a spectator sport." (Pollak, 1968, p. 78) He talks of ossifying mathematical formulation by starting with ready-made formulas, and that, as a result, students are deprived of finding for themselves the best mathematics possible for the situation, that is, of mathematizing the situation. (Pollak, 1968, p. 26) Servais, while admitting that it is easier to discuss the difficulties associated with the teaching of mathematics than to do something about them, states that the teaching of mathematics can no longer be an indoctrination. "L'enseignement . . . doit conduire à agir mathématiquement plutôt que do proposer à la contemplation une mathématique toute faite. (Servais, 1968, p. 43) Freudenthal (Freudenthal, 1968) consistently emphasizes that students must learn the process of mathematizing reality or of mathematizing mathematics. Revuz suggests that the teacher is not, or should not be, a disseminator of truths which the students must accept. He finds it deplorable that the almost universal concept of teaching is to divide the course into a more or less dogmatic exposé given by the teacher, and a series of exercises illustrating this dogma to be done by the pupils. (Revuz, 1968, p. 32)

The common belief at this Colloquium, was that mathematics is a process, not a conclusion, and that we must attach greater importance to what the child does in order to acquire certain concepts and facts. The emphasis is on activity on the part of the child in order to mathematize a given situation.

The remainder of this study will focus on the implementation of a specific type of discovery learning in a classroom, that is, the mathematizing mode. Descriptions and discussions of these experiences will focus on how the pupils arrived at their hypotheses and conclusions, and the teacher behaviour which helped bring this about.

CHAPTER III

THE DESIGN OF THE STUDY

I. Introduction

The purpose of this chapter is to state explicitly the aims of this study, and to describe in detail how these aims will be achieved.

II. The Problem

The purpose of this study was:

1. to develop a unit of study for the Mathematics 10 program using the mathematizing mode;

2. to describe the unit as taught to a group of grade ten students in sufficient detail so as to provide guidance to any teacher wanting to teach the same unit;

3. to analyze and discuss selected aspects of these classroom experiences of the unit so as to provide the reader with a deeper insight into the procedures and content of the unit;

4. to extract some general principles regarding the implementation of the mathematizing mode, and where possible, to compare them to the generalizations made by Johnston (Johnston, 1968) concerning the mathematizing mode.

III. The Instructional Setting

Nine lessons, each eighty minutes long were taught to a group of grade ten Mathematics 10 students at St. Joseph Composite High School by Dr. S. E. Sigurdson.

The Students

Seventeen students from an existing Mathematics 10 class of thirty-two students at St. Joseph Composite High School volunteered to be part of this study. Before volunteers were asked for, the teacher had excluded the chronic "skippers." The resulting group consisted of four boys and thirteen girls. According to the Cumulative Records of the school, their I.Q.'s as rated by the most recent tests (mostly from the preceding year) ranged from 82 (a girl who was fairly new in Canada) to 126 (a boy who had scored as high as 142 on previous tests). The mean I.Q. was 109. No one was repeating the course. Their grade nine marks in mathematics ranged from 45% to 90% and the mean was 65%.

The Physical Setting

Except for one lesson, the students were taught in the Television Studio at St. Joseph Composite High School. Two cameras were used to video-tape the lessons with one camera focussing on the teacher at all times. The decision as to what should be filmed was made by the television crew. Generally all of the teacher's behaviour and comments were filmed. They attempted to focus on individual students who were participating actively at the time. A microphone

was suspended over the students, and the teacher wore a lapel microphone at all times.

In order to accustom the students to the cameras, one full eighty minute period was spent at the outset of the unit, on a supplementary lesson which is not included in this study.

The Teacher

The teacher, Dr. S. E. Sigurdson, developed the mathematizing mode and has done a great deal in defining and refining this method. In a pilot study carried out at Harry Ainlay Composite High School, Edmonton, he developed and tested to his satisfaction a unit suitable for the Mathematics 10 course. From this emerged a set of exercises which very closely resembled the ones used for this unit.

The Content

The topic of this unit was factoring. This choice was not based on any consideration other than convenience - the students were ready to begin this chapter at the time of this study.

The lessons taught to the students were intended to cover the material found in Chapter IV in *Secondary School Mathematics, Grade Ten*, MacLean, Mumford, et al. This was the authorized text from the Department of Education.

The topics to be covered as outlined in Chapter IV of this text were:

4.1 The degree of a polynomial

4.2 General polynomials in one variable

4.3 A number and its factors

4.4 Factoring a polynomial whose terms have a common factor

4.5 Factoring after grouping to obtain a common factor

4.6 Factoring trinomials of the form $x^2 + bx + c$

4.7 Factoring trinomials of the form $ax^2 + bx + c$

4.8 Factoring a trinomial square

4.9 Factoring the difference of the squares of two numbers

4.10 Factoring after grouping to obtain the difference of the squares of two numbers

4.11 The quadratic equation

4.12 Factoring an incomplete trinomial square

4.13 The factor theorem

4.14 Summary

(MacLean, Mumford, 1965, p. IX)

Notice that the techniques of factoring are presented independently from any application of factoring. This is the traditional approach. Section 4.11 is the only section which deals with the solving of quadratic equations by use of factoring. This concept of using factoring to solve equations is not reviewed in the summary exercises in section 4.14.

Instead of presenting the theory of factoring independently from its application, as is done in the text, it was decided to present factoring as a tool for solving equations. Thus the general purpose of this unit was to develop efficient methods for solving certain equations, and the skill of factoring was developed for this end.

Six sets of problems were presented to the pupils in contrast to the sixteen sets of "practice exercises" found in the text. The first three sets of exercises were developed ahead of time, the other three were partially developed on the basis of the needs of the students. These exercises are included in the following chapter where they will be discussed in detail.

Two exams on factoring were given, one immediately after the unit was taught, and the other after two periods of traditional or expository teaching. The main purpose of these exams was to provide the regular teacher with information concerning her students' ability to factor. The results were used in this study only to rank the students in terms of present achievement and to compare these rankings with other rankings as described in the next section.

IV. The Procedure

This section deals with how the aims of this study were accomplished.

1. Aim: To create a unit of study for the Mathematic 10 program using the mathematizing mode; to teach it to a group of grade ten students; and to record these classroom experiences.

Procedure: This was accomplished by Dr. S. E. Sigurdson who developed and taught the lessons; by a Television Arts Craft class which video-taped them; and by the writer who observed and recorded them. Dr. Sigurdson was assisted by Miss Halia Boychuk and the writer during the pilot study.

2. Aim: To describe the unit as taught to a group of students.

Procedure: The writer observed the live lessons and attempted to record much of the teacher or pupil behavior which was not filmed. On the basis of these observations, numerous viewings of the video-tapes, and, examinations of the exercises, a description of the unit as taught to a group of students is given in Chapter IV. This description includes a discussion of the aims of the unit, the content and its organization, the specific purpose of each exercise, and teacher and pupil behaviour.

3. Aim: To analyze and discuss selected aspects of these classroom experiences.

Procedure: In Chapter V, the writer will

(i) discuss each day's lesson attempting to draw attention to crucial teacher and pupil behaviour, and organization of content. Some justification for the behaviour and organization as given in the current literature is presented.

(ii) analyze the participation of the students in teacher-led discussions.

In order to analyze the extent of the students' participation in class discussion, the writer categorized all parts of the lessons as seen on the video-tapes as either personal inquiry sessions or discussion sessions as defined in Chapter I. During a well-defined discussion session, each time a pupil spoke the writer placed a checkmark beside his name. The total number of checkmarks beside each name then represented the total number of different times each person spoke.

In an effort to discover whether or not all of the students were contributing through comments during personal-inquiry sessions, the writer re-viewed the video-tapes of two random personal-inquiry sessions from different days, and placed a checkmark beside the pupil's name if he spoke at least once, either to other students, or to the teacher.

The students were ranked according to: (1) the number of different times they spoke during the discussion sessions, (2) their marks on an exam given immediately after the unit, (3) their marks on an exam given three days after the first exam, (4) their intelligence quotient scores as indicated in their school records, (5) their mathematics marks on their last report card, (6) the amount they participated in regular class as rated by their regular teacher.

Using the Spearman Rank Correlation Coefficient the writer attempted to discover if there was a significant correlation between the participation of the students in the discussion sessions and (1) their participation in regular class, (2) their marks on the first and second exams, (3) their I.Q.'s.

(iii) attempt to classify the classroom experiences in terms of stages as outlined by Johnston. (Johnston, 1968)

Johnston's description of the mathematizing mode in terms of four stages is reviewed in Chapter V. The writer viewed the video-tapes with the intention of classifying the experiences in terms of these stages. This was repeated several times to check on the writer's consistency in the classification. Two other people were

also asked to attempt the classification in order to check the consistency of their classifications with that of the writer.

4. Aim: To extract some general principles regarding the implementation of the mathematizing mode.

Procedure: On the basis of the analysis and description in Chapter IV and V, the writer attempted, in Chapter VI, to redefine existing principles or generate new principles concerning the mathematizing mode in the areas of:

(i) the expectations of the teacher,

(ii) the general structure and organization of the lessons and exercises,

(iii) the participation of the students in teacher-led discussions,

(iv) the classification of the classroom experiences in terms of Johnston's stages.

Whenever possible, these general principles were compared to general statements made by Johnston.

CHAPTER IV

THE MATHEMATIZING MODE IN PRACTICE

I. Introduction

As a prerequisite to constructing a valid sequence of theoretical statements about the mathematizing mode, it was deemed necessary to begin by applying the method in an actual classroom situation. Davis calls for the following:

> What we seek . . . is a general description of what goes on in the classroom and in the school, from which we can begin to identify those variables which appear to be most decisive in determining success or failure in the long run . . . (Davis, 1964, p. 157)

A general description of the mathematizing mode as it was used in a specific classroom situation will be given in this chapter. A specially prepared unit was taught to a group of grade ten students. The content of this unit was described in Chapter III. Chapter IV will include the general objectives of the mathematizing mode and of the unit taught, the specific purpose of each set of exercises, the exercises themselves, and a general description of the teacher's behaviour and the pupils' reaction.

In the following chapter, selected aspects of these classroom experiences will be discussed and analyzed in depth with the intent of identifying crucial behaviour and techniques. In the final chapter, on the basis of these discussions, a series of generalizations about the mathematizing mode will be made, and these will be compared to Johnston's theoretical statements.

II. The Objectives of the Unit

Although this study is not concerned with outlining the objectives for the teaching of mathematics, it seems logical to provide the reader with a brief overview of the general objectives of the mathematizing mode. No formal attempt will be made to determine if these objectives were achieved.

The general objectives of the method then are:

1. To have the student develop his ability to create mathematics. Sigurdson says:

> If mathematics is created by hypothesizing, evaluating, and rejecting or accepting, and many mathematicians would agree that it is, then practice in these activities can only improve the student's ability to handle mathematics creatively. (Sigurdson, 1970, p. 133)

2. To create an atmosphere in which the student feels free to think about mathematics and to express his mathematical ideas. Biggs states as a principal aim of education "to free students, however young or old, to think for themselves." (Biggs, 1969, p. 3)

3. To have the student develop certain concepts, skills, and techniques in mathematics.

More specifically it was hoped that the students would be provided with the opportunity to:

1. propose ideas in the form of questions or answers;
2. defend their ideas;
3. alter their ideas when necessary;
4. make use of other people's ideas;
5. verify their own and other people's ideas;

6. discover applications or uses for their ideas;
7. discover existing patterns or relationships;
8. make use of their intuition or untested ideas;
9. make up symbols or vocabulary;
10. realize the value of common symbols or vocabulary.

In terms of the mathematical content covered in this unit it was hoped that the student would

1. understand the concept represented by the terms such as polynomial, coefficient, term, constant term, variable term, equation, equations of the first degree, . . . equations of the n'th degree, factoring, distribution, et cetera. The term itself would not be emphasized as much as the concept itself. At least in the first parts of the unit, the students would be allowed to label the concepts differently.

2. be able to use the following methods of factoring:

EXPRESSIONS	METHODS OF FACTORING
Binomials	
$ax + bx$	Common factor
$a^2 - b^2$	Difference of squares
$a^3 - b^3$	Difference of cubes (Factor Theorem)
$a^3 + b^3$	Sum of cubes (Factor Theorem)
$a^4 + 4b^4$	Incomplete square
Trinomials	
$ax + ay + az$	Common factor
$x^2 + bx + c$ $ax^2 + bx + c$ }	Inspection
$a^2 + 2ab + b^2$ $a^2 - 2ab + b^2$ }	Trinomial square

EXPRESSIONS	METHODS OF FACTORING
Trinomials (continued)	
$a^4 + a^2b^2 + b^4$	Incomplete square
Trinomials of the third degree in the variable	Factor Theorem
Polynomials of four or more terms	
$ax + ay + az + aw$	Common factor
$ax + ay + bx + by$	Grouping to obtain a common factor
$x^2 + 2xy + y^2 - z^2$	Grouping to obtain a difference of squares
Polynomials of the third degree in the variable	Factor Theorem

(MacLean, Mumford, 1964, p. 142)

3. recognize equations for which there are no real roots. (Such examples are not included in the text).

4. recognize equations whose roots are real and irrational. (Such examples are not included in the text).

5. be able to apply factoring to solve equations.

6. be able to realize when factoring is of use.

The question of introducing the "completing the square" method was left to be resolved on the basis of the needs of the students.

III. Objectives of the Exercises

In contrast to the sixteen sets of exercises provided in the text, only six were prepared for this unit. Each one was designed with a very specific purpose in mind. In this section, each set of exercises will be given along with a brief statement of its purpose Notice that each set is labeled as "Exercise # . . ."

Exercise #1

1. $x^2 + 3x + 2 = 0$
2. $x^2 - 4x = 0$
3. $x^2 - 4 = 0$
4. $2x^2 + 5x - 12 = 0$
5. $x^3 + 8 = 0$
6. $x^2 + 5x + 6 = 0$
7. $x^2 = 0$
8. $x^2 + x = 0$
9. $x^2 - 12 = 0$
10. $x^2 + 7x + x + 7 = 0$
11. $124x^2 = 0$
12. $x^2 + 10x + 25 = 0$
13. $2x(x - 7) = 0$
14. $2x^2 - 10x - 28 = 0$
15. $x^2 + 2x + 12x + 24 = 0$
16. $12x^2 = 48$
17. $x^2 + 4x - 6 = 0$
18. $x^2 - 9x + 20 = 0$
19. $x^2 - 14x + 49 = 0$
20. $100x^2 - 200x = 0$

The purpose of this exercise is to have the students discover methods of solving equations with one, two, or three terms. Hopefully the students would eventually discover factoring as a tool to solve these equations. Having done so, they would then have to develop techniques for using the following methods of factoring: (a) common factor, (b) difference of squares, (c) inspection of trinomials, (d) trinomial square (see page 40). The example of a sum of cubes, that is, $x^3 + 8 = 0$, is included only for the purpose of having the students realize that it was different from the others. The answer of -2 found by trial and error would be accepted.

Exercise #2

a) $x^2 + 9 = 0$

b) $2x^2 + 7x = 15$

c) $x^2 - x - 56 = 0$

d) $4x^2 - 100 = 0$

e) $4x^2 - 28x = 0$

f) $x^2 + 12x + 20 = 0$

g) $x^2 + 4x + 2 = 0$

h) $x^2 + 12x + 36 = 0$

i) $x^2 + 7x + 12 = 0$

j) $x^2 + 12x - 28 = 0$

k) $x^2 - 12x + 34 = 0$

l) $3x^2 - 2x - 8 = 0$

m) $x^2 + 2x + 1 = 0$

The purpose of Exercise #2 is the same as for Exercise #1. However, for those students who have discovered factoring as a tool, this exercise would provide them with more examples so that they can test their hypothesis, refine it if necessary, and provide more practice in factoring. In these two exercises, the students have the opportunity to discover that even with factoring, some equations are very difficult to solve, and others have no real solution. This would provide a need for a more efficient method of solution, namely the "quadratic formula."

Exercise #3

A. 2, 3

B. -3, 4

C. 2, 8

D. 1, 50

E. -4, -5

F. +1

G. 0, 12

H. -3, +3

I. 1/2, 8

J. 3/4, 1/4

The students were asked to find the quadratic equations whose roots are listed in this exercise. The purpose of this exercise is to reinforce the concept of factoring.

Exercise #4

1. $x^3 + x - x^2 - 1 = 0$

2. $1 + bx + b + x = 0$

3. $a(x - 2) - (x - 2) = 0$

4. $x^2 - q^2 = 0$

5. $(p - q)(2 - x) = 0$

6. $a(x + y) + (x + y) = 0$

7. $2 - x - 2y + xy = 0$
8. $x^2 - 2x + 4 = 0$
9. $x^2 + x - x^3 - 1 = 0$
10. $x^3 - 4x^2 - x + 4 = 0$
11. $x^3 + 5x^2 - 4x - 20 = 0$
12. $a(x - y) + 6(y - x) = 0$
13. $x^2 - 2ax + a^2 = 0$
14. $x^2 + ax - 2a^2 = 0$
15. $x^2 + k^2 = 0$
16. $6x^2 + 23x + 20 = 0$
17. $18t^2 + 15t + 3 = 0$
18. $3x^2n - 12n = 0$
19. $9x^2 + 30x = 25$
20. $8y^2 - 21 - 2y = 0$

The main purpose of this exercise is to have the students develop a technique for factoring polynomials of four terms, some with literal coefficients. Again practice is provided in using the methods of factoring with which they are familiar, but generally these are of a more difficult nature. For example, the difference of squares example uses a letter instead of a number: $x^2 - q^2 = 0$; $8y^2 - 21 - 2y = 0$ is not in the standard order; $(p - q)(2 - x)$ requires the students to realize the difference between the variable x and the variables p and q. As a matter of fact, the main difficulty anticipated in this exercise is to have the students realize the functions of the variable "x", the numerical constants, and the literal constants.

Exercise #5

(1) $x^2 - 100$

(2) $(x + 1)^2 - 49$

(3) $x^2 - 6x + 9 - y^2$

(4) $16a^4 - 8a^2 - b^4 + 1$

(5) $a^4 - 2a^2 + 1 - b^2$

(6) $a^2 - 10a + 25 - m^2$

(7) $9a^2 - 25b^2 + 24ac + 16c^2$

The purpose of this exercise is to have the students discover a method of factoring a polynomial of four terms by grouping to

obtain a difference of squares. This is the only very directed exercise and the only one which provides no direct practice in all the skills already learned.

Exercise #6

Factor each polynomial completely.

1. $x^2 + 4x + 3$
2. $9x^2 - 2$
3. $x^2 - y^2 + 4x + 4$
4. $x^2 + 10x + 22$
5. $12x^2 - 3$
6. $6r^2 - 13r + 6$
7. $2ax^2 + 11ax + 12a$
8. $x^2 - 7ax + 12a^2$
9. $m^3 - 6m^2 + 8m$
10. $(x - 4)(6x + 9) - (x - 4)(2x + 1)$
11. $t^4 - 16$
12. $x^2 - 6$
13. $y^2 - 4y + 1$
14. $x^2 + 1$
15. $8c^4 - 10c^3 + 3c^2$
16. $t^2 + 12t = 2$
17. $t^2 - 9t + 20$
18. $x^4 + 2x^2 - 15$
19. $y^2 - 3y$
20. $2t^4 + 9t^2 + 10$
21. $4y^2 - 3y - 1$
22. $40 + 65a - 3a^2$
23. $3x - 27x^3$
24. $x^2 + a^2 - 1 - a^2x^2$
25. $x^4 + 64$
26. $ax^2 - a^2x + as$
27. $4x^3 - 16x^2 + 13x - 3$
28. $64x^4 + y^4$
29. $x^4 - 3x^2 - 4$
30. $(x - a) + (ax - a^2)$
31. $25a^2 - 89a^2b^2 + 16b^4$
32. $(a - 1)^2 + 2 - 2a$
33. $a^2 - ab - a + b$
34. $a^4 + 5a^2b^2 + 9b^4$
35. $12x^3 + 16x^2 - 3x$
36. $x^2 - x - 6$
37. $2x^2 + x - 1$
38. $x^4 - 3x^2 + 1$
39. $6y^2 - 3y - 9$
40. $a^5 - 81a$

The purpose of this exercise is to provide practice in the methods of factoring as described on page 40.

IV. A Description of the Unit on a Day to Day Basis

A brief summary of the experiences of each day will be presented in this section. A more detailed discussion and analysis will be given in Chapter V.

The writer observed the classroom experiences which took place over a nine day period, and then viewed the video-tapes of these experiences a total of four times over a three month period. The description of this unit does not include details of sessions in which the teacher and students were involved in (1) verifying computations, (2) approximating answers by trial and error. The actual dialogue is included a few times only. (The reader is invited to examine the video-tapes to hear the actual dialogue.)

The purpose of this section is to provide the reader with a general view of the unit as it was implemented in the classroom. Teacher-pupil discussions are summarized, and the exercises are discussed very briefly.

Day 1

The actual problem of the unit, that of solving equations, was presented to the students, first of all in the form of a problem concerning dimensions, and then in an exercise containing a number of equations to be solved. The problem was as follows:

> A man is building a fence around a rectangular field. One side is 30 feet longer than the other, and the area is 90 square feet. Wnat are the dimensions?

Using a trial and error technique, the teacher with the help of the pupils found an answer. Exercise #1 was then assigned with instructions to find a value for x that "works". During this general introduction which lasted about 15 minutes, the teacher dominated the scene. The only comments made by the pupils were in the form of short answers to direct questions.

Following a work period, some of the questions from Exercise #1 were discussed. The following hypotheses emerged and were listed on the board.

Alex: There must be some definite way of getting the answers instead of just guessing.

Bob: (a) There are two answers for each question except . . .

John: To solve equations of the type $100x^2 - 200x = 0$, simplify as much as possible.

Judy: Zero cannot be an answer if there is a number alone.

Bob: (b) The left side of the equation can be written as factors.

None of these hypotheses were evaluated. The students were instructed to do more of the problems while keeping in mind the various hypotheses that had been made in order to weed out the useful from the useless ones.

After another period in which the students worked on their own, the teacher again led a discussion. The first point to come up was one of clarification. One student was attempting to use John's hypothesis on the problem $2x^2 - 10x - 28 = 0$. Some of the students pointed out that it did not apply.

Penny, using the example $x^2 + 3x + 2 = 0$, illustrated that the factors were $(x + 2)(x + 1)$ and that the answers were "the inverses of the numbers in the factors," that is, -2 and -1. A discussion revolved around verification of her work and the matter

was dropped. The teacher then ended the class by having the pupils tell him the answers to questions 1 to 10 which he put on the board.

A student came to him after class and suggested that it was probably just coincidence that the answers were the inverses of the numbers in the factors.

Day 2

The students continued to work on Exercise #1. The teacher suggested that there were two points of interest: (1) solving the equation, (2) finding out if the hypotheses are (a) correct, (b) useful. The discussion began with an attempt to solve $36x^2 + 5x = 0$. The students seemed to have completely forgotten about the method which they had evolved the previous day for solving such a question. Eventually, however, this problem and others like it were solved.

The teacher then asked the class to find examples which have one answer only. $x^3 = 8$, $x^2 = 0$, $124x^2 = 0$ were suggested. In a fairly directed session the teacher attempted to have the students consider the implication $x^2 = 0 \longrightarrow 24x^2 = 0$, and its converse $24x^2 = 0 \longrightarrow x^2 = 0$.

It was not until the second half of the class that the idea of factoring was again raised by Bob. Penny attempted once more to explain how it works, but was again unable to defend her arguments. Frank picked up the idea and gave a lengthy explanation as to how factoring could be used. His hypothesis was: "Factor and take the inverse of the number." The teacher tried to have the students focus on why the hypothesis was correct and eventually a few of the students correctly stated the reason.

The teacher briefly summed up and turned his attention to the perfect square trinomial. Judy suggested that if the factors are both the same, then there is one answer only. This was added to Bob's first hypothesis. The hypotheses being considered now were:

Alex: There must be some definite way of getting the answers instead of just guessing.

Bob: (a) There are two answers for each question except
(1) for examples such as $x^3 = 8$, $x^2 = 0$, $124x^2 = 0$
(2) when both factors are the same.

John: To solve equations of the type $100x^2 - 200x = 0$, simplify as much as possible.

Judy: Zero cannot be an answer if there is a number alone.

Bob: (b) The left side of the equation can be written as factors.

Frank: To solve these equations, factor and take the inverse of the number.

Day 3

The students continued to work on Exercise #1 during the first half of the class. The first problem to be discussed was one which didn't factor: $x^2 + 4x - 6 = 0$. The teacher had various students suggest replacements for the -6 so that it would factor.

A summing up of Exercise #1 was accomplished by having the students categorize the questions according to those with one answer and those with two answers. The questions were all corrected in the process.

Exercise #2 was handed out with the very brief instructions to continue to find an x that works. This exercise contained the same types of equations as those in Exercise #1, as well as two trinomials in which the coefficient of x^2 was other than 1, and an equation for which there was no real solution.

The students soon came upon a problem which clearly indicated that their hypothesis to "factor and take the inverse of the numbers" was not adequate. They factored $3x^2 - 2x - 8 = 0$ into $(3x + 4)(x - 2) = 0$, and suggested that the answers were -3, -4, and +2. One student clearly defined the problem, a new rule was made up, and after being confronted by the teacher with certain examples, the students finally came to realize why their rule worked. The teacher summarized their discoveries.

Day 4

Exercise #2 was corrected while having the students categorize the problems into the categories of equations with (a) one answer, (b) two answers, (c) no answers, and (d) hard-to-find answers. Most of the time was spent in obtaining answers, by trial and error, to two questions which did not factor easily. In the first case, once both of the answers were found, the teacher asked for the factors. In the second case, one answer only was found, and the question as to whether or not a second answer did exist, and if so, what it was, was left open to further discussion.

Exercise #3 was handed out with the instructions "Here are the answers. Find the questions." A brief discussion followed in which the term "quadratic equation" was reviewed and the students were told to find the quadratic equations whose roots were listed in Exercise #3.

While working on Exercise #3, some and eventually all of the students went back to finding a second answer to $x^2 - 12x + 34 = 0$, the question from Exercise #2 to which they had found one answer

only. The teacher finally had the students "discover" that once they had one answer, the second answer could be easily found by division or subtraction rather than by trial and error.

The teacher then took about ten minutes to teach the students by a direct question and answer technique, how to solve the equation by completing the square. The last few minutes were spent in doing others problems from Exercises #1 and #2 by this method.

Day 5

The students were assigned Exercise #4 and a discussion followed as to the difference between the variables b and x in a question such as $1 + bx + x + b = 0$. Some of the students worked on this exercise while others put the answers to Exercise #3 on the board. Although the students were not anxious to correct any mistakes on the board, with some urging from the teacher, they eventually did. The second half of the class was spent in discussing various questions from Exercise #4.

Day 6

The students worked on Exercise #4 for the whole period. Discussion sessions were often indistinguishable from private inquiry sessions as there was almost constant dialogue between the teacher and some of the pupils. Solving equations by completing the square was reviewed, and some time was spent on learning how to complete the square.

Day 7

The teacher asked that the students find different examples from Exercises #1 to #4 of expressions with 1, 2, 3, and 4 terms, and that they organize these examples into groups which could be factored in similar ways. As a result, many different types of factoring were reviewed. During the second half of the class, the students were given seven expressions to factor (Exercise #5), all of which could be done by using a "difference of squares" method. For those who finished early, the teacher provided questions such as $a^4 + a^2 + 1$ which would be factored by using both the "completing the square" method, and the "difference of squares" method.

Day 8

Exercise #6 was assigned. This was meant to be a straight-forward practice session but since the exercise contained many new ideas and many "trick" or difficult questions, the students ended up exploring many old ideas in greater depth and were exposed to many new ideas.

Day 9

The students continued to work on Exercise #6. A great many basic concepts were reviewed and some good practice in factoring was provided. Unfortunately, however, there was not enough time to complete the exercise nor to summarize the findings.

V. Summary

In order to provide the reader with a more accurate picture of what took place, an overall summary is included in Tables I and II on the next two pages.

A few points are worthy of note:

1. Each class period was eighty minutes long.

2. The unit was completed in nine such class periods.

3. Only six sets of exercises were provided.

4. The time allowed for each set of exercises was not artifically restrained by the length of the class periods. Exercise #1 for example initiated activities which took up two and one-half class periods.

5. There were three opportunities for categorizing problems. These were used as "summing-up" activities but also served to initiate the following activity.

6. There was a great deal of movement back and forth through the exercises. Thus, one activity was never isolated from the others.

7. All exercises except Exercise #5 provided practice for skills already acquired and reinforcement for concepts already discussed.

8. All exercises were related to the following exercise.

In general, while working on Exercise #1, the students simply explored the problem and attempted to provide solutions. In Exercise #2, they hypothesized, tested their hypotheses, and modified them as necessary. Exercise #3 was basically one of reinforcement of concepts already learned. In Exercises #4, #5, and #6, students acquired new techniques of factoring while practicing the ones they had already acquired.

Table I

SUMMARY OF ACTIVITIES OF DAY 1 TO DAY 4

Day*	Activity	Sub-activity	Purpose
1	Exercise #1	Introduction Exercise #1	To discover factoring as a tool for solving equations and to develop these methods of factoring: (a) common factor (b) difference of squares (c) inspection of trinomials (d) trinomial square
2			
3	(200 minutes)	Categorizing problems of Exercise #1	
	Exercise #2 (80 minutes)	Exercise #2 Categorizing problems of Exercise #2	
4	Exercise #3 (40 minutes)	Exercise #3 Short expository lesson on "completing the square" method More work on Exercise #1 and #2	

*Each day represents an 80 minute period.

Table II

SUMMARY OF ACTIVITIES OF DAY 5 TO DAY 9

Day	Activity	Sub-activity	Purpose
5 6	Exercise #4 (200 minutes)	Exercise #4 Correct Exercise #3 Categorize problems from Exercises #1 - #4	To develop techniques for factoring polynomials of four terms some with literal coefficients.
7	Exercise #5 (40 minutes)	Exercise #5	To develop technique of factoring by grouping to obtain "difference of squares"
8 9	Exercise #6 (160 minutes)	Exercise #6 Constant reference back to other exercises	To provide practice in different methods of factoring

CHAPTER V

AN ANALYSIS OF THE MATHEMATIZING MODE IN PRACTICE

I. Introduction

A factual account of the mathematizing mode as applied in a specific situation has been given in the previous chapter, as well as an outline of objectives of the unit and exercises. In this chapter a more detailed discussion of these experiences together with some theory supporting the behavicur and organization will be given. An analysis of student behaviour in terms of participation in group discussion will be presented, and finally an attempt will be made to classify the experiences according to categories as outlined by Johnston. (Johnston, 1968)

This and the previous chapter will provide the reader with a clear picture of the mathematizing mode as used in the classroom and an insight into the philosophy, goals, and expectations which govern the teacher's behaviour.

On the basis of this analysis, a sequence of generalizations will be made in Chapter VI. These will be compared to Johnston's theoretical description of the mathematizing mode.

II. A Discussion of the Unit on a Day to Day Basis

In Chapter IV, a brief summary of each day's experiences was given. Here, the teacher and pupil behaviour will be discussed in

greater detail, and some justification for certain techniques as given in the current literature will be given. The object of this is to provide a clearer picture of the mathematizing mode in practice.

Day 1

Exercise #1 (Table III) was organized so that the student, in quest of a method of solving quadratic equations, would discover factoring as a tool. The underlying belief here is that all mathematical concepts and techniques can be presented as reasonable answers to reasonable questions. Davis supports this in saying:

> All of the paraphernalia of science or mathematics--concepts, equipment, data, techniques, even attitudes and expectations--arise out of the act of tackling problems and arise out of inquiry. We want the concepts which the students form to arise in this same way. (Davis, 1964, p. 148)

And so, rather than starting with the technique of factoring and working up to what is thought of as the highest level of learning, that is, problem-solving, the learner began with problem-solving and had to work his way through the necessary concepts and techniques needed to solve it. The students had been asked to find values for "x" which would satisfy the given equations - thus they would discover factoring as a tool to accomplish this.

The central theme of the entire unit was presented in the first lesson, and the concepts most fundamental to the whole unit were there to be discovered in that first exercise as well as in succeeding exercises. This is in sharp contrast to the more traditional approach of presenting techniques bit by bit, and having

TABLE III

Exercise #1

1.	$x^2 + 3x + 2 = 0$	11.	$124x^2 = 0$
2.	$x^2 - 4x = 0$	12.	$x^2 + 10x + 25 = 0$
3.	$x^2 - 4 = 0$	13.	$2x\ (x - 7) = 0$
4.	$2x^2 + 5x - 12 = 0$	14.	$2x^2 - 10x - 28 = 0$
5.	$x^3 + 8 = 0$	15.	$x^2 + 2x + 12x + 24 = 0$
6.	$x^2 + 5x + 6 = 0$	16.	$12x^2 = 48$
7.	$x^2 = 0$	17.	$x^2 + 4x - 6 = 0$
8.	$x^2 + x = 0$	18.	$x^2 - 9x + 20 = 0$
9.	$x^2 - 12 = 0$	19.	$x^2 - 14x + 49 = 0$
10.	$x^2 + 7x + x + 7 = 0$	20.	$100x^2 - 200x = 0$

the students perfect the skills involved before letting them know what they can be used for. Here, it was hoped that the students would realize that factoring would help them solve equations long before they would attempt to perfect their skill in factoring. A possible advantage would be that the students would have a definite mathematical need for perfecting a skill.

The exercise was such that all the students could relate to it at some level. On the one hand, the students could, if nothing else, obtain an answer by trial and error. At the other extreme, they could develop the quadratic formula. Between these two extremes, they could employ a variety of types of factoring. And although the exercise was not ordered in any way, such as from easiest to most difficult, there did exist a great range in terms of difficulty. For example, all students were able to solve $x^2 = 0$, and find at least one answer to $x^2 - 4 = 0$ without any difficulty, but a large number of students were greatly challenged by $2x^2 + 5x - 12 = 0$ and $x^2 + 4x - 6 = 0$. In other words there was something in this exercise for everyone. Sigurdson refers to this as making the initial problem a "primitive" one. He says ". . . the problem should be approachable by the slowest pupils in the class and yet still be full of potential for the best students." (Sigurdson, 1970, p. 81)

The equation $x^2 + 30x - 90 = 0$ was developed by the teacher and pupils on the basis of the information provided in the original problem given by the teacher. The pupils suggested various values of "x" which might make the equation true and finally a fairly close

approximation was found. Some of the students, very eager to imitate the teacher, continued to use this technique long after the more efficient method using factoring had been discovered and discussed. Had these students themselves, rather than the teacher, proposed finding a solution by a trial and error technique, perhaps they would not have focussed excessively on it later.

Exercise #1 contained twenty questions all of which, except one, were equations to the second degree with one, two, three, and four terms. It was brought out in discussion that the equation $x^3 + 8 = 0$ was an exception in this exercise. The nineteen questions then represented essentially the only types of problems, in terms of the factoring involved, which would be considered during the next few days. No plan of attack was suggested by the teacher - the instructions were: "Find an x that works."

Not until very late in the unit did the teacher ask the pupils to focus on questions of the teacher's choice. The students were largely responsible for the direction which the discussion took.

As a direct result of discussing the answers to a few problems, the students soon made their first discovery that there were sometimes two answers. This developed into Bob's hypothesis that all of these problems had two answers. The pupils immediately evaluated this hypothesis by finding an exception, and the hypothesis was changed to "There are two answers to each question except . . . ." Thus, it was no longer a neat hypothesis which was either right or wrong, but rather it presented a whole new aspect of the problem to

be investigated. Wherever possible, the students' ideas were not considered right or wrong, but rather as statements to be qualified.

At this point, Alex complained "Mostly we're just guessing - isn't there some certain set process that we can go through to get the answer?" The teacher encouraged this train of thought by telling them that it would be most useful to investigate this matter. And so, another part of the problem was revealed, that is, not simply to get an answer, but to find a more efficient way to get it. Sigurdson and Johnston, in their description of student behaviour patterns state: "The students must be given the mind set that they are to continually search for short cuts, patterns, formulae, and generalizations as solutions to problems." (Sigurdson and Johnston, 1970, p. 131) Many of the students were expressing their frustration with the trial and error method and were thus well disposed towards finding a more efficient method.

The students then focussed on $100x^2 - 200x = 0$ and others like it. Two hypotheses emerged as a result of doing a few examples. One was concerned with a method of solving these by dividing out a common factor and the other had to do with realizing that "0" was an answer here, but that it could not be an answer if there was a constant. The students had grouped together similar problems, and had made astute observations. Although John used the word "simplify"

rather than "divide out the common factors" the class understood what he meant and the teacher did not insist on different terminology. The emphasis was on correct concepts not vocabulary.

The teacher did not reinforce this method which the students had discovered and they seemed to have forgotten about it by the next day. (see Chapter IV, p. 48) Does this contradict Bruner's claim that discovering things for oneself is an aid to memory? (Bruner, 1960) Beberman points out

> . . . a precise verbalization from the student is not a sure sign that he is aware of the class of instances of the generalization, for he may regard the generalization itself as just another "instance." (Beberman, 1958, p. 27)

Beberman then would suggest that the students had not really discovered a general method which worked for a certain class of problems, but rather that they had managed to solve a few specific problems. No real generalization had been made.

Bob then suggested that the left side of the equation could be rewritten as factors and gave a few examples to illustrate his hypothesis. He did not indicate how this would help to solve the equations. The hypothesis was recorded on the blackboard by the teacher and the only student comment was "I can't see any advantage to factoring - it's just a waste of time." The hypothesis nevertheless remained and the teacher suggested that more thought be given to it and the other hypothesis. Bob's hunch was further developed next day by Penny and given the proper formalism and computational correctness by Frank. At this point, however, it was allowed to stand as an incomplete but possibly relevant idea. Bruner says:

> Intuitive thinking, the training of hunches, is a much-neglected and essential feature of productive thinking not only in formal academic disciplines but also in everyday life. The shrewd guess, the fertile hypothesis, the courageous leap to a tentative conclusion - these are the most valuable coin of the thinker at work.
> (Bruner, 1960, pp. 13-14)

And again he emphasizes "it is the intuitive mode . . . that yields hypothesis quickly, that hits on combinations of ideas before their worth is known." (Bruner, 1960, p. 60) By not demanding that Bob justify his hypothesis and by allowing it to remain for the students' consideration, the teacher was clearly encouraging the students to air their hunches.

A little later, the idea of factoring was picked up by Penny who clearly illustrated its use in solving equations. Surprisingly enough, except for a discussion concerning the verification of the factors, little was said about her explanation. She did not have the conviction to really defend her ideas, and perhaps since it had not come from the teacher, the others shrugged it off as coincidence. It should be remembered that one of the objectives of this method is to have students realize that mathematics can come from themselves and each other as well as from the teacher or text.

The teacher kept returning the responsibility of producing mathematics to the class by turning their questions back to them, by refusing to evaluate their comments or suggestions, and simply by not showing them how to do the problems more efficiently. Occasionally he introduced conventional vocabulary.

Day 2

Penny wanted to know whether or not "x" must or can be zero in the type of problem in which there is no constant. This led to the question as to whether zero was the only answer. The teacher asked for an example to work with and was given $36x^2 + 5x = 0$. Note that the question is difficult to solve by trial and error. The students gave no indication whatever of realizing that they had worked out a method for doing such a problem the day before and were at no point reminded of this by the teacher.

The teacher then changed the question to $36x^2 + 6x = 0$. The idea caught on immediately and Bob suggested that $x^2 - x = 0$ was very similar and that it had two answers, 0 and 1. Other students followed example, giving similar problems to which they knew the answers. This finally led to Bob's suggestion of using the method which had been worked out the day before. At the teacher's suggestion, the students made up more similar examples and worked them out. Thus the technique was reinforced. Finally the original problem was re-tackled and solved.

Polya supports the technique of making up similar problems as an aid in solving a problem. He suggests: "If other means fail, we should try to imagine an analagous problem." (Polya, 1957, p. 182) Heinke defines the method of variation as

> ". . . a process of changing elements of the data, or conclusion, or both, . . . which has been proved to be true or is accepted as true with a view to obtaining a new set of data, or a new conclusion, or both, resulting in a new statement." (Heinke, 1957, p. 148)

Thus, in contrast to Polya, variation is used more to produce new problems. The pupils, following the teacher's example, used the method of variation in many ways. They used it to discover solutions, to verify hypotheses, and to expand their hypotheses.

In the discussion of $x^2 = 0$, Alex explained that x had to be zero because either a positive or negative number would yield a number greater than zero. He also gave a very good explanation as to why $24x^2 = 0$ if $x^2 = 0$. No one, however, seemed to grasp the significance of the converse, or for that matter, to realize that the converse was different from the original statement. The teacher attempted to draw this out of the pupils but gave up when he realized that they were simply not ready for it. Bob's hypothesis was then expanded into "you get two answers except: 1) when you have a cube, 2) examples of the type $x^2 = 0$, $24x^2 = 0$, etc."

In order to keep the pupils from merely trying to get a correct answer, the teacher asked the students to identify the questions which had one answer and those which had two answers. When the idea of factoring came up again, Jerry suggested that we now have a definite way of knowing for certain whether or not there were one or two answers. Unfortunately this important idea was completely lost when he applied it incorrectly to $x^2 - 4 = 0$ and said that there was one answer only.

Bob suggested factoring once more and Penny again tried to explain how it worked. Again no one really listened to her. Frank, however, was more convincing and using the example $x^2 + 8x + 7 = 0$, forced the class to see how this could be solved by factoring. The

class was delighted with this "neat" little technique and no one seemed interested in finding out why it worked. When asked to verify their solutions, no one thought of substituting in the factored form. The teacher persisted and asked if $(x + 1)(x + 7) = 0$, why is $x = -1$, $x = -7$? Eventually a few students suggested that if a product was 0, then one or the other factor, or both, had to be zero. The fact that no one seemed to know this the next day is again a good indication that a precise verbalization of concept is no indication that any understanding exists. (Beberman, 1958, p. 27) The teacher briefly reviewed the concepts which had just been discussed.

The discussion then turned again to problems with one answer only and Bob suggested $x^2 - 14x + 49 = 0$ as an example. The teacher told the class that this was called a perfect square trinomial and asked for other examples. Many other examples were given by the pupils. As a result, Bob's hypothesis was then further expanded to include the perfect square trinomial as a type with one answer only.

Day 3

The students focussed, over and over again, on expressions which would not factor. In this case, the question at hand was to find a solution for $x^2 + 4x - 6 = 0$. Some of the students had found an approximate answer by substituting various values for "x" and were thus able to tell the rest of the class that the problem did have a solution. The teacher briefly considered their solutions and then

asked how the constant in the problem could be changed so as to make the trinomial factorable. The students suggested five different possibilities. Again they were using the method of variation so favoured by Heinke. (Heinke, 1957, pp. 146-54)

Some review and practice was provided while the students categorized the problems from Exercise #1 into types with one or two answers. For example, when the students gave ± 3.45 as a solution to $x^2 - 12 = 0$, the teacher asked how this could be done by factoring. (Many students were still falling back on a trial and error method). Both $(x + 3.45)(x - 3.45) = 0$ and $(x + \sqrt{12})(x - \sqrt{12}) = 0$ were suggested. The questions with one answer only were further categorized according to the exceptions listed in Bob's hypothesis.

At this point, the students had found answers to all of the questions, either by trial and error, or by factoring. And, although more could have been drawn from these questions, the students were not inclined to spend any more time on them. And so, the teacher handed out a new set of problems, Exercise #2 (Table IV), essentially of the same type as those in Exercise #1, again not ordered in any particular way. Depending on the students, these questions could provide straightforward practice, or they could provide the necessary feedback for the pupils still testing their unproven ideas.

TABLE IV

Exercise #2

a) $x^2 + 9 = 0$	h) $x^2 + 12x + 36 = 0$
b) $2x^2 + 7x = 15$	i) $x^2 + 7x + 12 = 0$
c) $x^2 - x - 56 = 0$	j) $x^2 + 12x - 28 = 0$
d) $4x^2 - 100 = 0$	k) $x^2 - 12x + 34 = 0$
e) $4x^2 - 28x = 0$	l) $3x^2 - 2x - 8 = 0$
f) $x^2 + 12x + 20 = 0$	m) $x^2 + 2x + 1 = 0$
g) $x^2 + 4x + 2 = 0$	

The students soon pointed out the difference between the question $x^2 + 9 = 0$ which has no real solution, and $x^2 + 4x + 2 = 0$ which has a solution not easily found by factoring. Some time was spent trying to solve the latter question by trial and error. Once an answer had been found, the students did not know whether or not another one existed, and could not think of finding another solution by any method other than trial and error.

So far the students had been applying their rule to "factor and take the inverses of the numbers" without any difficulty. Now they stumbled upon mathematics which made them stop and think about their original hypothesis, and eventually, refine it. The problem was to find a solution for $3x^2 - 2x - 8 = 0$. After factoring it into $(3x + 4)(x - 2) = 0$, the students suggested that the answers were

-3, -4, +2. The teacher simply suggested that they investigate more thoroughly and went on to discuss another question. At this point, Betty, who was obviously annoyed and frustrated, clearly and emphatically defined the problem at hand. Only then did the whole class stop to consider it. After factoring $2x^2 + 7x - 15 = 0$ into $(2x - 3)(x + 5) = 0$, she explained that using the inverses of -3 and +5, that is +3 and -5, one simply did not get the right answers. Judy suggested that the +3 should be divided by +2 but could not explain why. The students now had a new rule which worked, and which they had made without help from the teacher, but were still unable to understand why it worked. The teacher then gave them the equation $(205x - 37)(x + 1{,}000{,}500) = 0$ and asked for a solution. $\frac{37}{205}$ and -1,000,500 were suggested. He then asked that they verify their solutions. Amid moans and groans at the thought of first multiplying out these factors, the students stumbled on the idea that each of these numbers made one of the factors equal to zero, and that the product was then necessarily zero. Their cries of "Yeah!" and "Neat!" indicated, at least in Hendrix's terms,* that discovery had taken place.** It is interesting to note that the

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*With reference to knowing when a student has made a discovery, Hendrix speaks of "A sudden start, a flush of excitement, and a student begins writing answers almost as fact as he can put them down." (Hendrix, 1961, p. 297)

**Unfortunately this was not recorded on the video-tape due to some technical difficulties.

required concept had been explicitly stated by the teacher and several students on the previous day and yet it was not until the students got the appropriate feedback from the mathematics itself that they gave any indication of understanding. Trivett says "It is not a human mode that learners automatically learn by being told. Humans do not pay much attention to what others are saying." (Trivett, 1970, p. 11)

Day 4

The review was again in the form of having the students categorize the problems, and while this was done, many difficulties with factoring were cleared up. Thus, factoring was again being treated simply as a technique which would help in the solution of greater problems, rather than an end in itself. The groundwork for Exercise #3 was established when the teacher asked the pupils for the factors of an equation to which they had found a solution by trial and error. The equation was $x^2 + 4x + 2 = 0$ and the answers were supposedly -3.45 and $-.54$. The factors suggested were $(x + \sqrt{2})(x + \sqrt{2})$ and $(x + 3.45)(x + .54)$. The teacher then had the pupils explain why the first answer was incorrect. Again Heinke's method of variation was used when the teacher asked how the original equation could be changed in order to make this answer correct. (Heinke, 1957, pp. 146-54) Time was then spent on consideration of the second answer.

Exercise #3 (Table V), was designed to reinforce the basic concepts developed in Exercises #1 and #2, and yet was not a practice session in factoring. The roots of the equations were given

and the students were to find the original equations. The students had nearly finished the exercise when they were challenged by the teacher to go back and find a second solution to the equation $x^2 - 12x + 34 = 0$ from Exercise #2. Notice that this question had been left with one solution only. Although the students had approximated one answer as + 4.51, they spent a great deal of time in trying to find a second answer also by trial and error. Finally the teacher asked what the factors should be. The answers given by various students were, in this order:

$$(x \quad\quad) (x \quad\quad) = 0$$
$$(x - 4.51) (x \quad\quad) = 0$$
$$(x - 4.51) (x - \quad\quad) = 0$$

And suddenly the students "discovered" a method for finding the answer. A key question from the teacher had led the students away from inefficiently using the trial and error method.

TABLE V

Exercise #3

A.	2, 3	F.	+1
B.	-3, 4	G.	0, 12
C.	2, 8	H.	-3, +3
D.	1, 50	I.	1/2, 8
E.	-4, -5	J.	3/4, 1/4

Traditionally, the teacher asks most of the questions. Here, the pupils were doing most of the asking. They now began to express great dissatisfaction with using a trial and error method on equations which would not factor. They expressed a strong need for a more efficient method. There was absolutely no reason for not giving them such a method at this point. They knew precisely what their difficulty was and what was needed. And so, the teacher demonstrated in a fairly direct way, but making full use of any knowledge that could contribute, how to solve the quadratic equation by completing the square. To do this, the teacher used the same question as above, that is $x^2 - 12x + 34 = 0$ and began by asking the students how the question could be varied to make it very simple to factor. Note the use again of the method of variation, perhaps more in the terms defined by Polya. (Polya, 1957, p. 182) $x^2 - 12x + 36 = 2$ was suggested, and there was no problem in finishing up with $(x - 6)^2 = 2$, and $x - 6 = \pm \sqrt{2}$ and $x = 6 \pm \sqrt{2}$. Judy explained that it was simply a matter of rewriting it as a perfect square. Thus, when the students demonstrated a clear need and readiness for a concept, as they did here, the teacher did not hesitate to use a very direct approach.

Now the students went back to Exercises #1 and #2 to do various questions using this method. Answers which had been approximated by trial and error were now found precisely using this new method. The students had a more complete picture. Davis says "We learn by successive approximations, and there is no final or absolutely perfect ultimate version in any of our minds." (Davis, 1964, p. 152) And again he states:

> ". . . an orientation based upon the notion of gradual modification of the individual's internal cognitive structure appears to us as highly appropriate for studying the learning of mathematics." (Davis, 1964, p. 152)

Certainly so far, the methods of solving the equations had been worked out gradually. Now the students moved freely through Exercises #1, #2, and #3, filling in all the gaps.

Day 5

After handing out Exercise #4 (Table VI), the teacher asked that the answers to Exercise #3 be put on the board. Some of these answers although incorrect, were not corrected by the students until much later. At no time did the teacher himself point out any errors but he did ask the students twice to check over their work. Lila had given the equation $x^2 - 12x = 0$ as the one whose roots were 0 and 12, but had given her intermediate work for it as $(x - \sqrt{12})(x + \sqrt{12}) = 0$. Alex suggested that something was wrong, but as soon as he was asked to state what it was, he backed down and said that he had found his mistake! It should be remembered that one of the aims in teaching here was to have students develop confidence in their own ability to create mathematics. Alex did not have this confidence yet. When Judy pointed out the error, Alex ignored it all as best he could. The question was corrected and a few more simple errors were pointed out by the pupils.

During this time the students were also working on Exercise #4. This exercise afforded more practice in factoring and thereby solving equations similar to those in Exercises #1 and #2. As well, it contained many equations with four terms, often containing letters

TABLE VI

Exercise #4

1.	$x^3 + x - x^2 - 1 = 0$	11.	$x^3 + 5x^2 - 4x - 20 = 0$
2.	$1 + bx + b + x = 0$	12.	$a\ (x - y) + b\ (y - x) = 0$
3.	$a\ (x - 2) - (x - 2) = 0$	13.	$x^2 - 2ax + a^2 = 0$
4.	$x^2 - q^2 = 0$	14.	$x^2 + ax - 2a^2 = 0$
5.	$(p - q)\ (2 - x) = 0$	15.	$x^2 + k^2 = 0$
6.	$a\ (x + y) + (\ x + y) = 0$	16.	$6x^2 + 23x + 20 = 0$
7.	$2 - x - 2y + xy = 0$	17.	$18t^2 + 15t + 3 = 0$
8.	$x^2 - 2x + 4 = 0$	18.	$3x^2n - 12n = 0$
9.	$x^2 + x - x^3 - 1 = 0$	19.	$9x^2 + 30x = 25$
10.	$x^3 - 4x^2 - x + 4 = 0$	20.	$8y^2 - 21 - 26 = 0$

instead of numbers as constants. These were more easily factored if they were first of all grouped in pairs. The purpose of this exercise was to provide more practice and more challenge in factoring quadratics - some of the problems were fairly difficult. Also the intent was to expose the students to a new type of factoring. The new work was in no way separated from the previous work. It was up to the student to discover the familiar problems and to decide which ones would probably require new techniques. Although most of the problems were fairly difficult, there did exist still a fairly good range so that all of the students could find something which they could do.

The difficulties that emerged were many. Alex wanted to rewrite $1 + bx + x + b = 0$ as $1 + 2bx = 0$. Time was spent by the teacher with the class deciding upon equivalent expressions of the original equation. Eventually, Judy factored it correctly into $(b + 1)(x + 1) = 0$ and a lengthy discussion ensued as to whether or not $b = -1$ as well as $x = -1$ could be accepted as an answer. The teacher insisted that the problem was to find x only and some of the students insisted that $b = -1$ also made the statement true, as it of course did. This discussion was repeated with reference to the question $(p - q)(2 - x) = 0$, with the students insisting that $p = q$ and $x = 2$ were both answers. They pointed out that x could be anything if $p = q$. The teacher replied "We're only interested in x," but the students felt that they were in fact discussing x. The discussion ended only when the teacher definitely stated that only x would be discussed. This was probably wise since such difficulties

are often best resolved over a period of time rather than at the moment.

Quite a lot of time was spent in straightforward verification which was, to a large extent, the result of Frank contradicting Penny every time she suggested something. Penny was starting to become more eloquent in defending her ideas.

A few of the students, Betty in particular, wanted to know how these expressions were being factored in the first place. The only suggestion given to her, by the other students, was to do so by trial and error.

Day 6

This class was essentially a work period to do Exercise #4, in which discussion sprang up often and spontaneously. The following are descriptions of interesting points and problems brought up by and encountered by the students.

After trying to find a solution, by trial and error, to $x^2 - 2x + 4 = 0$, many students used the "completing the square" method. They seemed very pleased to discover that it could be rewritten as $(x - 1)^2 = -3$ and that this obviously had no solution. Lorraine commented "At least now we know that we shouldn't keep trying to look for an answer."

The students were still factoring polynomials of 4 terms by straight trial and error and they seemed totally unaware that there existed certain techniques which could be of use to them. The teacher, in a direct manner, finally showed them how to group in pairs and then factor. For the first time the teacher himself

suggested which questions should be discussed. The discussion centered mostly around factoring and not the solutions of the equations. For example $(x^2 - 4)(x + 5) = 0$ was left in that form, and when Frank pointed out that there were three answers to this question, no one paid much attention. Most of the students had written down +4 and -5 as the answers.

Without changing the form of $a(x-y) + b(y-x) = 0$, Greg gave the answer as $x = y$. The teacher asked that the left hand side be factored, and with the help of Penny who suggested that +b be changed to -b, this was done.

Lucy suggested that $x^2 + k^2 = 0$ is true only if $x = k = 0$. The teacher replied that the equation would then disappear.

Some students experienced difficulty in factoring $6x^2 + 23x + 20 = 0$. Others helped out. A few were still experiencing difficulty in stating the solution even after they had the correct factors.

More difficulties were encountered in solving $x^2 - 2ax + a^2 = 0$, first of all in factoring it, then in deciding what x should be to make the statement true.

Judy solved $3x^2n - 12x = 0$ just by looking at it.

Judy asked the teacher how to complete the square. Quite a few students were having difficulties with this and so, using the following questions, the teacher tried to get the students to discover how to do it.

$x^2 + 10x +$ __________

$x^2 + 6x +$ __________

$x^2 +$ __________ $x + 121$

$x^2 + 5x +$ __________

$x^2 +$ __________ $x + 20$

Unfortunately, only five of the students gave all of the answers, and only Bob was able to get the last question correct. No formal rule was ever stated. The teacher then had the students solve the equation $x^2 + 5x + 6 = 0$ by completing the square. This was the first time in which they used this method on a question which factored easily.

No one really seemed to know how to solve $9x^2 + 30x - 25 = 0$ until John suggested that it be rewritten as $(3x + 5)^2 = 50$.

This lesson was, at times, totally teacher directed. The pupils were by now well acquainted with the problem at hand, and both they and the teacher were fully aware of most of their difficulties. They all needed help with some very specific aspects of the exercise and the most sensible way of helping them was by having the teacher give a very directed lesson. The teacher worked with individuals, or with small groups of students. Private discussions developed into class discussions very naturally and the students worked a great deal with each other. The prevailing mood seemed to be that of enjoyment in their pursuit of solutions, alone and with each other.

Day 7

The students had previously categorized the equations in terms of those with one, two, or no solutions. Now they were asked to categorize them as those with one term, two terms, three terms, and four terms. (see Chapter IV, p. 52) No very specific explanations were given as to its exact meaning, however, definitions were evolved as the students' answers were discussed. The following description is again of the interesting points and

problems brought up by the students, and of the teacher's reaction to them.

Alex suggested that $(x - 1)^2$ be placed under the category of one term, but the rest of the class decided that it had three terms. Lila suggested that $12x^2 = 48$ be considered as a one term example but changed her mind when the teacher rewrote it as $12x^2 - 48 = 0$ and compared it to $x^2 - 4 = 0$. More examples of this type were suggested and factored by the pupils. The teacher told them that it is called "difference of squares" and asked why this is so. Bob gave a clumsy explanation which the teacher patched up. Bob suggested that $x^2 - 4x = 0$ be rewritten as $(x - 0)(x - 4) = 0$ and Lucy said that this was equivalent to $x(x - 4) = 0$. Lila suggested that $x^2 + 4 = 0$ be solved by completing the square but Frank talked her out of it. Bob suggested that $x^2 - q^2 = 0$ was a different type and John argued that it should be categorized as having one term since it could be rewritten as $x^2 = q^2$. The teacher pointed out that it was a "difference of squares" type and Alice solved it. Lila gave $(x - q)(p + 2)$ as an example of a type with two terms, but Frank explained that it had four terms. Bob argued that $x^2 + k^2 = 0$ was different from $x^2 + 4 = 0$ since the former had a solution, that is, $x = k = 0$, whereas the latter example had no solution. The students grouped $x^2 + 4x + 2 = 0$ and $9x^2 + 30x - 25 = 0$ together because both were difficult to solve. They pointed out that the perfect square trinomials have one answer whereas the others have two. A few examples of equations with four terms were factored, mostly by the teacher with the help of the students, using the method

of first grouping in pairs. John again suggested $x^2 = x$ as an example of one term problems, but Lorraine showed how it could be rewritten as $x^2 - x = 0$ and Lucy factored it.

Judy, using the idea of grouping those with four terms into pairs, suggested that $x^2 + 7x + 12 = 0$ be rewritten as $x^2 + 4x + 3x + 12 = 0$, or $(x^2 + 4x) + (3x + 12) = 0$ before factoring into $x (x + 4) + 3 (x + 4) = (x + 4) (x + 3) = 0$. The teacher then suggested that the original $x^2 + 7x + 12 = 0$ be rewritten as $x^2 + 5x + 2x + 12 = 0$ and the students had to explain why this would not be helpful. Lucy and John argued about the usefulness of this method in terms of saving time.

The students then, in categorizing the problems from the previous exercises, re-examined their concepts about them, obtained more practice in factoring, and had a great deal of freedom in proposing new ideas. The teacher gave very direct help at times, and encouraged the students to take responsibility of verifying and changing notions where necessary. At times he too gave incorrect suggestions which had to be argued against by the pupils.

Exercise #5 (Table VII), was assigned to the students. This exercise was structured so as to suggest to the students that certain polynomials of four terms could be grouped so as to factor using the "difference of squares" method. These were not given in the form of equations to be solved, but merely as expressions to be factored.

Most of the students expanded $(x + 1)^2 - 49$ and then factored it. The teacher showed them how to factor it directly as a "difference of squares." $x^2 - 6x + 9 - y^2$ was then quickly done by

most students. Frank, however, had to be told where $(x - 3)$ came from. The only suggestion given by the teacher to factor $16a^4 - 8a^2 - b^4 + 1$ was that its order was incorrect. The students themselves found that $9a^2 - 25b^2 + 4ac + 16c^2$ should have been $9a^2 - 25b^2 + 24ac + 16c^2$ and the correction was made.

TABLE VII

Exercise #5

(1)	$x^2 - 100$	(5)	$a^4 - 2a^2 + 1 - b^2$
(2)	$(x + 1)^2 - 49$	(6)	$a^2 - 10a + 25 - m^2$
(3)	$x^2 - 6x + 9 - y^2$	(7)	$9a^2 - 25b^2 + 24ac + 16c^2$
(4)	$16a^4 - 8a^2 - b^4 + 1$		

Because a few of the students were finished, the teacher challenged them with factoring $a^4 + a^2 + 1$. John immediately pointed out that it should have $2a^2$ and Bob suggested that it be rewritten as $a^4 + 2a^2 + 1 - a^2$. The rest was easy - so much so, that Bob repeated happily five times "Ah, this is <u>so</u> easy!" A few more examples were completed.

Once again the students at all levels had been challenged. A new type of factoring had been introduced in a fairly directed way, that is, the exercise was tightly structured to bring about very specific objectives. This is in contrast to the loose structure of the earlier exercises. And, this new knowledge was used to launch a few of the students on to a new type of factoring.

Day 8

Again this lesson could be characterized as a work period with a great deal of discussion coming up spontaneously. In no way was it a straightforward practice session with the students doing many similar examples. Instead, they were given Exercise #6 (Table VIII), containing all types of factoring, varying from very simple to extremely difficult. The exercise was not ordered in any way. It is most difficult to report on this session, because the students were working together or in small groups, very animatedly, on different questions. The teacher discussed various questions with these groups when they wanted to. Some of these discussions developed into full class discussions. The following description focusses on a few of these discussions with the hope of illustrating the very open thinking displayed by the pupils.

After discussing how to factor $a^4 + a^2 + 1$ with the class, the teacher asked that they factor $x^4 + 5x^2 + 9$. Here is part of the resulting discussion.

Bob: Write it as $x^4 + 6x^2 + 9 - x^2$

Lucy: That's the same as $(x^2 + 3)^2 - x^2$

Alex: That factors into $(x + 3 + x)\ (x + 3 - x)$

Frank: No! That's $(x^2 + 3 + x)\ (x^2 + 3 - x)$

Bob: What happens if we have $x^4 + 7x^2 + 9$?

John: That gives $x^4 + 6x^2 + 9 + x^2$

The class agrees that this does not factor.

Bob: Write it as $x^4 + 8x^2 + 16 - x^2 - 7$.

That's $(x^2 + 4)^2 - \sqrt{(x^2 + 7)}^2$

TABLE VIII

Exercise #6

Factor each polynomial completely.

1. $x^2 + 4x + 3$
2. $9x^2 - 2$
3. $x^2 - y^2 + 4x + 4$
4. $x^2 + 10x + 22$
5. $12x^2 - 3$
6. $6r^2 - 13r + 6$
7. $2ax^2 + 11ax + 12a$
8. $x^2 - 7ax + 12a^2$
9. $m^3 - 6m^2 + 8m$
10. $(x - 4)(6x + 9) - x - 4)(2x + 1)$
11. $t^4 - 16$
12. $x^2 - 6$
13. $y^2 - 4y + 1$
14. $x^2 + 1$
15. $8c^4 - 10c^3 + 3c^2$
16. $t^2 + 12t = 2$
17. $t^2 - 9t + 20$
18. $x^4 + 2x^2 - 15$
19. $y^2 - 3y$
20. $2t^4 + 9t^2 + 10$
21. $4y^2 - 3y - 1$
22. $40 \div 65a - 3a^2$
23. $3x - 27x^3$
24. $x^2 + a^2 - 1 - a^2x^2$
25. $x^4 + 64$
26. $ax^2 - a^2x + as$
27. $4x^3 - 16x^2 + 13x - 3$
28. $64x^4 + y^4$
29. $x^4 - 3x^2 - 4$
30. $(x - a) + (ax - a^2)$
31. $25a^2 - 89a^2b^2 + 16b^4$
32. $(a - 1)^2 + 2 - 2a$
33. $a^2 - ab - a + b$
34. $a^4 + 5a^2b^2 + 9b^4$
35. $12x^3 + 16x^2 - 3x$
36. $x^2 - x - 6$
37. $2x^2 + x - 1$
38. $x^4 - 3x^2 + 1$
39. $6y^2 - 3y - 9$
40. $a^5 - 81a$

Teacher: That's brilliant! But why not just leave the middle term and change the last term?

John: That would be $x^4 + 7x^2 + (3.5) - 4.25$

In another example Bob admitted that $x^2 + 1 = 0$ had no solution, but pointed out that it can be rewritten as $x^2 + 2x + 1 - 2x$ or $(x + 1 + \sqrt{2x})\ (x + 1 - \sqrt{2x})$.

A discussion followed as to why this was not useful. The teacher and pupils attempted to establish informal rules which would determine when factoring was appropriate.

Most of the students expanded and collected like terms before factoring $(x - 4)(6x + 9) - (x - 4)(2x + 1)$, rather than factoring directly. The teacher showed them how to factor it directly.

A few more questions were discussed as a group, but most of the action was among the pupils themselves. Unfortunately the writer made no attempt to record this, and it came out very badly on the video-tape. That is, the numerous conversations are indistinguishable one from the other.

Day 9

The mood which had prevailed throughout the lessons was broken. The students were told that they would have to write an exam the next day and they suddenly became very interested in learning set ways of doing things. This class was interesting especially in its contrast to the previous day's class. The students were very subdued and there was very little exchange between the students. The exchange between the teacher and pupils was mainly

the result of the students asking how a question should be done.

Some fairly good review took place but there was unfortunately not enough time to summarize the findings.

It is hoped that the description of these lessons has provided the reader with some very explicit information about the mathematizing mode in a practical classroom situation as well as an insight into the mood which was thereby created.

III. The Behaviour of the Students in Terms of Participation in Discussion Sessions

Background

The purpose of this section is to provide a general description of student behaviour in terms of participation in discussion sessions, that is, in teacher-led discussions.

All of the parts of the lessons from the first seven days were categorized as either personal-inquiry sessions or discussion sessions as defined in Chapter I. During the last two days the writer found the personal-inquiry sessions to be indistinguishable from the discussion sessions since a private discussion between the teacher and a pupil often developed into a group discussion, or as quickly, became a private discussion again.

The students were then ranked according to the number of times which they spoke during the discussion sessions. No attempt was made to tally the number of times that students spoke to each other or to the teacher during personal-inquiry sessions. However, an attempt was made to determine whether or not each student spoke

at least once either to another student or to the teacher during two different personal-inquiry sessions.

As well as being ranked according to the number of times which they spoke during the discussion sessions, the students were also ranked according to (1) their marks on an exam given immediately after the unit, (2) their marks on an exam given three days after the first exam, (3) their intelligence quotient scores as indicated in the school files, (5) their mathematics marks on the last report card, (6) the amount they participated in regular class as rated by their teacher.

The Results

The results were as follows:

1. During the first seven classes approximately two-fifths of the time was spent in personal-inquiry sessions and the remainder of the time in discussion sessions.

2. Four of the seventeen students made slightly over one-half of the comments during the discussion sessions. (Table IX)

3. The four boys in the group ranked in the top five of the group in terms of making comments during discussion sessions.

4. All but one of the seventeen students talked either to a classmate or to the teacher at least once during the two personal-inquiry sessions examined.

5. The rankings are shown on Table IX. Using the Spearman Rank Correlation Coefficient it was found that for these seventeen students, the correlation between:

TABLE IX

Student Rankings

Student	Number of Comments Made During Discussion Sessions	RANKINGS					
		Comments Made During Discussion Sessions	Participates in Regular Class	Marks on First Exam	Marks on Second Exam	Math Marks on Last Report Card	I.Q.
Bob	71	1	2	1	11	11	11
John	59	2	1		5	5	1
Judy	46	3	4	6	7	4	4
Frank	37	4	6	2	4	8	7
Alex	34	5	3	15	16	17	8
Penny	33	6	14	3	1	1	9
Bertha	26	7	5	9	12	6	6
Lila	21	8	8	7	2	3	5
Lucy	20	9	7	8	9	9	14
Harriet	15	10	9	16	10	16	15
Betty	13	11	10	13	8	7	10
Lorraine	12	12	11	5	6	10	3
Alice	7	13	16	12	13	12	13
Leona	7	14	12	11	17	15	16
Virgina	5	15	15	4	3	2	12
Cathy	1	16	13	10	15	14	17
Stephanie	0	17	17	14	14	13	2

(1) the students' participation in the discussion sessions in this unit, and their participation in their regular class as evaluated by the teacher was .866. Therefore at the $p<.01$ level, this is significant.

(2) the students' participation in the discussion sessions in this unit, and their marks on their first exam was .444. This is not significant.

(3) the students' participation in the discussion sessions in this unit, and their marks on the second exam was .340. This is not significant.

(4) the students' participation in the discussion sessions in this unit, and the I.Q.'s as reported on their school records was .373. This is not significant.

(5) the students' marks on their first exam, and their mathematics marks on their last report card was .647. This is significant at the $p<.01$ level.

(6) the students' marks on the first exam and their I.Q.'s as given by the school records was .02. This is not significant.

(7) the students' marks on their first exam and on their second exam was .640. This is significant at the $p<.01$ level.

6. The students, this time excluding the boys, were again ranked according to the number of times which they spoke during the discussion sessions, their marks on the first exam, and their I.Q.'s. Using the Spearman Rank Correlation Coefficient it was found that for these thirteen girls, the correlation between

(1) their participation in the discussion sessions in this unit, and their marks on the first exam was .401. This is not significant.

(2) their participation in the discussion sessions in this unit, and their I.Q.'s as reported on the school records was .264. This is not significant.

Discussion of the Results

In another study which utilized the mathematizing mode, one in which a unit on areas was presented to a group of grade seven pupils, the teacher found that approximately three-fifths of the time the students worked on their own or in groups with the rest of the time spent in class discussion. (Sigurdson, 1970, p. 88) In this study, the students worked approximately two-fifths of the time on their own or in groups.* Johnston does not suggest an ideal division of time between private-inquiry sessions and discussion sessions.

Certain students definitely dominated the discussion sessions. It is interesting to note that they are also the ones who dominated in their regular classes, and that the only four boys in the group ranked in the top five in terms of making comments during the discussion sessions. Also of interest is the fact that there was no significant correlation between the amount that the students participated in the discussion sessions, and either their marks or their I.Q.'s. This was true for the group as a whole and also for the girls alone.

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*Only seven out of nine lessons were analyzed for this purpose. This may have affected the results.

Their rankings in terms of marks on the first exam, the second exam, and on their last report card were consistent. There was no significant correlation between these marks and their I.Q's.

IV. A Classification of the Classroom Experiences

An attempt was made to classify the classroom experiences which were discussed in the previous section, according to the categories as outlined by Johnston in his description of the mathematizing mode. (Johnston, 1968) It was hoped that the unit described within such a framework (or a modification of it), together with the description of the unit from the previous section, would provide a very clear example of the mathematizing mode in practice.

Therefore, a review of Johnston's theoretical description of the mathematizing mode will be given, followed by a description of the attempt to use it to categorize the experiences described in this study. The results of this attempt will be given.

The Mathematizing Mode as Described by Johnston

In 1967-68, a project was conducted by five investigators from the Department of Secondary Education at the University of Alberta to compare discovery teaching to expository teaching on several dimensions. At that time, Johnston investigated the nature of discovery learning and helped evolve a particular discovery method which he labeled the "Mathematizing Mode." He described it in terms of four stages: 1) introduction of the activity and

exploration of the problem, 2) brainstorming session, 3) seminar type discussion, 4) summary. (Johnston, 1968, pp. 60-61) A brief review of the description of these stages follows.

During stage one, the problem is presented to the pupils within the broadest possible framework. The pupils then, on their own, explore the problem as fully as possible with the intention of presenting their ideas to the class in the form of hypotheses. "Student activity in the form of personal inquiry is the essential characteristic of stage one . . ." (Johnston, 1968, p. 44)

Stage two is characterized by pupil activity in the form of making hypotheses. The teacher who acts as moderator records all the hypotheses without evaluating them. The students are not expected to justify their ideas at this time, and are encouraged to express themselves in their own words. The teacher will sometimes attempt to delay a verbalization until he feels that the student really knows what is going on.

During a seminar type discussion, the students must evaluate their hypotheses, reinforce the correct ones, and unlearn those which are not correct or useful. This is called stage three. The teacher may introduce at this point, certain definitions and conventions. Practice is provided.

Finally, during the fourth stage, the ideas introduced earlier are expressed more precisely, formulas are considered, and the work summarized.

Briefly then, the four stages are:

1) Introduction and exploration (personal-inquiry)

2) Hypothesizing (discussion)

3) Evaluation of hypothesis and practice (discussion)

4) Formal summarization (more teacher directed)

Usually there is a cycling of the first three stages many times before the final stage is introduced.

A Classification of the Experiences

The writer felt that certain small but nevertheless very important changes had to be made in Johnston's description before it would be of any use in classifying the experiences of this study. As it stood, Johnston's four stages could not apply to the unit for the following reason: Johnston describes each stage as either a personal-inquiry session or a group session. The writer found that any of the four stages could be of either form. For example, the students, with the aid of the teacher, sometimes explored the general aspects of a problem before working on their own to hypothesize possible solutions. (For example, see Chapter IV, pp. 48-49). And, they were often asked by the teacher to try to evaluate the given hypotheses as they worked alone. (For example, see Chapter IV, pp. 47-48).

Also, the writer felt that the introduction was so completely different from the exploratory stage, that it merited a category of its own. It will be shown too, that the exploratory stages were not always preceded by an introductory stage and therefore should not be categorized together.

An attempt will be made to show that the following activities did take place during the lessons:

1. Introduction - The unit or particular parts of the unit are introduced by the teacher. This introduction is brief and leaves a lot for the students to define. The students are immediately set to work on a task which will reveal to them a general picture of the problem undertaken in the unit. The intent is to have the students find out what the problem is about by doing rather than by listening. This is in definite contrast to the usual practice of giving very precise, clear, and complete instructions.

2. Exploration of the Problem - The students, in attempting to find a solution to the problem when, as yet, they have no given method of doing so, redefine the problem, reorganize it, uncover existing relationships or other interesting aspects, and, in general become very familiar with it. On their own, or as a group they prepare themselves to formulate tentative solutions.

3. Hypothesizing - The student, on their own, consider possible solutions to the problem or possible relationships which may exist, and in group discussion, present their ideas to the class. They are encouraged to present to the class all of their ideas on the topic, whether or not these ideas seem directly useful in obtaining a solution. None of these ideas are considered right or wrong at this point. They are all considered as helpful in shedding light on the topic. After all, incorrect ideas are useful if one can show why they are incorrect, or if one can alter them to make them correct. (Another useful technique is to have the students alter the original problem so as to make the incorrect solution correct.)

4. Evaluation - The students, alone or with the teacher, evaluate their own, and other students' ideas, by checking them against mathematical reality. The given hypotheses may be simply accepted as true or rejected as false. More often, the hypotheses are altered by, for example, qualifying the statements. The same hypothesis can be evaluated several times, each time being elaborated upon as more information come to light. In this manner, closure may be forestalled as the same ideas are re-examined and new information is added.

5. Summary - The students alone, or more often the teacher with the help of the students, summarize the concepts and techniques arrived at, verbalize the generalizations more precisely, and learn the required conventions.

6. Practice - The students practice the techniques that they have learned, and practice using the concepts. This can take place in the form of a definite exercise after they have evaluated them or after the ideas have been summarized, or the students may be practicing throughout all of the activities, and especially during the evaluating stage as Johnston suggests.

This description is, in essence, the same as that given by Johnston. It differs in that stage one is now described in terms of two activities - introduction and exploration, and, a practice activity has been added. Also, no activity is described as being exclusively of the form of a personal-inquiry session, or of a group discussion. All activities are of either form.

An attempt will now be made to show that these activities

did exist in the actual classroom practice described in Chapter III and discussed in the previous section, and that they occurred more or less in the listed order.

An Application of the Classification

Can the activities of seventeen different students over a period of nine days be classified under six predetermined categories? It was finally decided to attempt such a classification on the basis of the remarks made by the teacher only. These remarks would, hopefully reflect the activities of the students. For example, if the teacher said: "While doing these problems, keep in mind these hypotheses, and try to see if they are 1) correct, 2), useful," (see Chapter IV, p. 47) then the activity was categorized as an evaluating activity. If the instructions were "Try to come up with anything that would help us solve these," the activity was categorized as a hypothesizing activity.

In order to make the classification, the writer observed the classroom experiences, and then, over a three month period, viewed each of the video-tapes of these lessons a minimum of four times. The teacher who taught these lessons viewed them once, and another teacher (not a mathematics teacher) also viewed them once. Each time, an attempt was made to categorize the various parts of the lesson according to the six previously described activities.

The results were as follows: There was agreement among the viewers insofar as classifying the activities of the first four days. However, the writer was unable to categorize the activities

of the next five days with any consistency, and there was no agreement with or between the other viewers.

The results of the classification of the activities for the first four days are given in Table X. Notice that ten minutes are missing from day 4. During the last ten minutes of the lesson, the teacher taught in a very direct and expository fashion. This could not be classified under any of the existing activities and was therefore omitted.

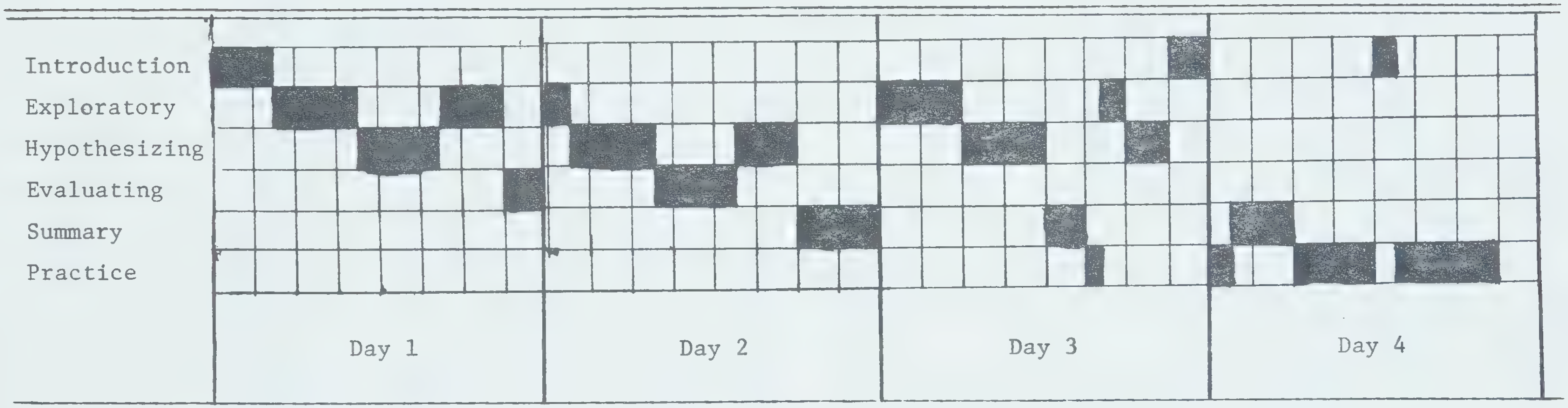
Discussion of the Results

The results, as shown in Table X, indicate that all six activities did take place during these four days, and more or less in the order outlined. Johnston suggests that there is often a cycling of the stages before the practice and summary stage takes place. (Johnston, 1968, p. 60) This is shown to be true here. Notice that approximately equal amounts of time were spent on all the activities except for the introductory activity. This is interesting since this is really the only teacher dominated activity, and here, a little less than one-twelfth of the total time was spent, as compared to less than one-fifth of the total time for each of the other five activities. (See Table XI)

More will be said about these results in Chapter VI.

TABLE X

A Classification of the Unit



Each block represents 10 minutes, each day represents 80 minutes.

TABLE XI

Summary of Classification

	Total Times
Introduction	25
Exploration	50
Hypothesizing	55
Evaluating	60
Summary	60
Practice	60
Total Minutes =	310

V. Summary and Conclusion

The purpose of this chapter was to provide a more detailed description of the mathematizing mode as applied in a particular classroom situation. In order to do this, each lesson was discussed in greater detail. An analysis of the behaviour of the students in terms of their participation in class discussion was given. And, finally, the activities of the first four lessons were classified according to six categories.

The implications of this discussion and analysis of the classroom experiences will be discussed in the next chapter.

CHAPTER VI

IMPLICATIONS, SUMMARY, AND CONCLUSIONS

I. Introduction

The purpose of this chapter is to discuss the implications of Chapter V. More specifically, an attempt will be made to extract the general characteristics of the lessons and exercises discussed in Chapter V, and to review the general expectations of the teacher as reflected in these lessons. Also, the writer will discuss the results of the analysis of the participation of the students in the discussion sessions, as well as the results of the classification of their activities. These findings will be directly compared to those of Johnston. (Johnston, 1968) The intent is to determine whether or not the description, or parts of this description, of the theoretical framework of the mathematizing mode as given by Johnston, must be modified, expanded, or rejected.

II. A Discussion of the Implications of Chapter V

The purpose of this section is to glean from the descriptions in Chapter V, the general characteristics of the mathematizing mode which can be of aid in applying the method more generally, and where possible to compare these characteristics to those described by Johnston. (Johnston, 1968)

As much as our actions are most often determined by our purposes, so too is the teacher's behaviour determined by his expectations. Therefore, on the basis of the teacher's behaviour as described in Chapter V, an attempt will be made to describe his expectations.

The general structure of the lessons and exercises with reference to specific example from Chapters IV and V will be given. It is hoped that this discussion too will reflect the teacher's expectations.

The writer will discuss the implications of the results of the analysis of student participation in class discussion, and the implications of the results of the classification of the class experiences.

Expectations of the Teacher

The intent here is to discuss the most general expectations of the teacher only, that is, the overall goals which determine the teacher's behaviour and attitudes. For a more specific list of the teacher's expectations, see Chapter IV, pp. 39 - 46.

Perhaps the most important belief which the teacher must hold is that students, young and old, can produce mathematics. Then the emphasis quickly changes from having the student imitate to having the student produce. There is no doubt that during the lessons described in Chapters IV and V, the students did produce mathematics. They found ways, several ways, of solving equations; they evolved many different methods of factoring; and they discovered

characteristics of different forms of equations such as the fact that the perfect square equation has one answer, some equations have no real solution, etc. That the teacher guided them and sometimes assisted them directly, that some of their conclusions were incorrect or incomplete and had to be rectified later, that their ideas were not necessarily original, makes it of no less value. The emphasis was on production of ideas rather than on imitation.

The teacher realized, from the start, that students, like all human beings, learn gradually, and therefore never attempted to produce the whole picture at once. Davis says "We learn by successive approximations, and there is no final or absolutely perfect 'ultimate version' in any of our minds." (Davis, 1964, p. 153) For example, the students had evolved a rule for solving $x^2 + 3x + 2 = 0$ and were quite certain that it worked for all quadratic equations. Only later did the teacher confront them with $3x^2 - 2x - 8 = 0$ and thereby forced them to expand their rule. (See Chapter IV, p. 50) Their picture was then more complete. He allowed them to struggle with a clumsy trial and error method for a long time on problems such as $x^2 + 4x + 2 = 0$ and only later did he help them evolve a method of "completing the square." (See Chapter IV, p. 51) He allowed them to factor expressions such as $x^2 - 6$ and $x - \sqrt{6}$ until they themselves realized where the usefulness of factoring ended. (See Chapter V, p. 84) Therefore the students sometimes left the classroom with incomplete or incorrect ideas, but, given time, they clarified and rectified them. John, for example, left

class one day certain that factoring would be of no aid in solving equations. (See Chapter IV, p. 48) He soon changed his mind. The teacher then must expect that his students will learn gradually. Some days may seem very productive, others will not. Johnston says:

> It is not possible to protect children from gathering wrong (partially correct) ideas; hopefully however, by a succession of approximations, these partially correct ideas will evolve into a more complete understanding of the mathematical concepts. (Johnston, 1968, p. 54)

The teacher must expect that different students may make different types of contribution.* For example, Bob proposed ideas, right or wrong, which he could not always justify immediately. Frank corrected other people's ideas and clearly restated their ideas. Betty brought the class to a grinding halt when they were ignoring very basic problems. Some of the students contributed nothing in class discussion but were very helpful to their classmates when working in a small group with them in terms of helping them evaluate and apply their ideas. The teacher then must try to have the class profit from these various contributions. Undoubtedly it is a team effort, but our concern is, of course, with the individual. The individual students were at different levels and therefore responded in different ways. The teacher expected this and tried to capitalize on it.

The most important beliefs which must guide the teacher then are:

1. Students can produce mathematics.
2. Students learn by gradual modification.
3. Students respond and therefore contribute in different ways.

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*An informal description of the characteristics and contributions of some of the pupils is given on pages 115-116.

General Characteristics of the Lessons and Exercises

How do we get students to produce mathematics? How do we capitalize on individual differences? How do we cope with the fact that human beings learn gradually? Very specific ways of accomplishing these goals will be discussed here with references to actual examples described in Chapters IV and V. It is hoped that this discussion will be of aid in applying the method more generally.

The following characteristics then seem peculiar to the general structure of the lessons and exercises in a unit taught using the mathematizing mode.

1. The theme or purpose of the entire unit is described in the first lesson. The concepts most fundamental to the unit are there to be discovered in the first lesson as well as in successive lessons. The student is aware of his task for the unit, although he may not be aware of the problems which will hamper him along the road to success. During this first lesson, the student must see as general an overview of the problem as possible - later he will fill in the gaps.

An important notion related to this is the idea that each concept or technique to be learned, is seen as fulfilling a purpose in a greater design. In this case, in the first lesson, the pupils were presented with the problem of solving equations. This was the central theme of the unit. Factoring then was taught as a tool to solve equations which was in turn taught as a tool to solve problems. The general picture was all there in the first lesson - the details would be filled in later.

Johnston is possibly speaking of the same thing when he says:

> . . . the problem (activity) should be stated within the broadest possible framework . . . should demand from the pupils an extension of an already known framework to "accommodate" for the new concepts. (Johnston, 1968, pp. 41-42)

2. The learner is presented with a problem. The entire unit is presented in the form of an initial problem which is to be solved. Bruner would have

> . . . the learners begin with problem-solving. Once confronted with a problem, whether embedded in the materials of instruction or directly presented by the teacher, the learner will be led to move back through the hierarchy to form the needed associations, attain the necessary concepts, and finally, derive the appropriate rules for solving the problem. (Shulman, 1970, p. 53)

This concept is closely associated with the first point. The students were presented with the problem of how to solve equations - a large number of concepts and techniques were then learned to solve this problem.

3. The exercises must be such that the pupils can relate to them at different levels. Exercise #4 (see Chapter IV, pp. 43 - 44) for example, was such that it could provide review, more practice in solving by trial and error, or, challenge the students to derive new techniques. Exercise #5 (see Chapter IV, p. 44) could help students learn a very specific method of factoring, or it could provide all the necessary background for a related type of factoring.

4. The students should have a hand in defining or redefining the problem. When the students were asked to categorize the examples into those with one, two, three, or four terms (see Chapter V, p. 78), explanations from the teacher were very limited.

Were all of the equations to be put first of all into a form equal to zero? What is a term? Was $(x - 3)(2x + 1) = 0$ considered one, two, three, or four terms? The definitions and requirements of the problem were soon evolved by the teacher with the class, after they had encountered the difficulties, not before. This supports Johnston's notion that:

> . . . by stating a problem in an incomplete way, and within a broad framework, opportunity is supplied for the students during the course of their exploration period to redefine and clarify the original problem. (Johnston, 1968, p. 43)

5. The material in each exercise is not ordered or grouped in any particular manner. The students are expected and encouraged to look over the various problems and if necessary, order them from easiest to most difficult, or more important, to group them with respect to their mathematical characteristics. The teacher can encourage this form of activity during the discussion sessions as an aid to solving problems or, as a form of review.

Exercise #4 (see Chapter IV, pp. 43 - 44) for example, contained new and old work. The students were expected to find out what was new work for themselves. Exercise #2 (see Chapter IV, pp. 42 - 44) contained some equations with no real solutions, others with irrational solutions - again in no particular order. The students were required to order it themselves.

6. The students can be helped to grasp the underlying structure of the mathematics by requiring them to categorize their work. This helps them to establish order in their ideas and specifically it helps them to discover general patterns or characteristics. Several times the

teacher asked the students to categorize the problem in terms of restrictions which he set out. (See Chapter V, p. 65 and p. 78) As a result, the students discovered more characteristics of the different types of polynomials. This technique was directly responsible for having the students isolate different methods of factoring. (e.g. factoring of trinomial squares, factoring of difference of squares, common factor, etc.) Categorizing the problems serves two purposes then. It helps the student to discover generalizations, and it helps him organize what he already knows in a more meaningful fashion.

7. All methods of solution must be encouraged. Johnston says:

> The teacher should accept different problems arising from the same situation . . . (Johnston, 1968, p. 119)

> The teacher should accept wrong or partially correct hypothesis with equal enthusiasm. (Johnston, 1968, p. 120)

and,

> The teacher must be willing to accept several methods of solution to the problem. (Johnston, 1968, p. 121)

The emphasis is on making use of all of the students' suggestions. Although the trial and error method was a very clumsy method to solve equations, the teacher encouraged the students to use it until they themselves realized that the method of factoring was more efficient. No attempt was made by the teacher to introduce a quick efficient method right from the start. When the students were finally very frustrated with using trial and error on equations which didn't factor, only then did the teacher introduce the method of completing the square. Certainly students must come to realize that rules, formulas, and symbolism are tools for more efficient,

effective, and sophisticated mathematical manipulations. The student must become acutely aware that patterns very often exist, patterns which may lead to more efficient methods of solution. Problems must not be done in isolation, rather they must be compared to each other to bring out the commonalities. Johnston emphasizes that "the students should exhibit a 'mind set' that mathematics means looking for best ways, short-cuts, patterns or generalizations in the solution of problems." (Johnston, 1968, p. 122) The important point here is that it is the mathematics itself, rather than the teacher that should reveal to the student the need for more efficient methods of solutions, short-cuts, and formulas.

8. The method of variation must be used by both teacher and pupils in order to help find solutions, and, just as important, to produce new problems so as to expand. Heinke says: "One of the fine achievements possible . . . is a renewal of the childhood disposition to initiate fruitful inquiry with the question 'What if . . . ?" (Heinke, 1957, p. 154) The students were able to find a method of solving $36x^2 + 6x = 0$ by considering similar examples to which they knew the answers (see Chapter V, p. 64), and Bob produced many new questions for consideration by simply altering part of another question. (For example, see Chapter V, p. 82) Perhaps this is the very essence of the mathematizing mode. After all, an important step in having the students produce mathematics is to have them formulate new questions. In this unit it was the teacher who initially set the example of using the method

of variation, and he encouraged the students at all times to use it.

9. The students must be encouraged to express their ideas or hypotheses, in their own words, whether or not they can justify them at the time, and whether or not they seem complete at the time. There are many important notions here.

First of all, the student must clearly communicate his ideas to his classmates, but he must not be distracted from the mathematics by focussing excessively on the words to describe it. There is time later on to learn the correct names of certain procedures. Initially we are concerned with the ideas and not with how they are presented.

Johnston supports this idea fully.

> . . . the students should be encouraged in the use of "intermediate language." A precise mathematical description should not be required until the concept involved has been incorporated into their cognitive structure. The students should realize that the "thing" exists by itself. A name is used to identify it, and a formula may be employed to find it quickly. Precision in the use of a language is an aid to remembering and discussing a useful concept. It should not be made a stumbling block to the learning process. (Johnston, 1968, pp. 46-47)

It is interesting to note that in the first few lessons the ideas or hypotheses as they were called, were carefully recorded by the teacher on the board (see Chapter IV, p.47), but in the later stages, such strict procedure was not followed. The very process of writing a statement on the board may stop the student's flow of ideas as he focuses on how to say it, rather than on what he was going to say. Clearly both procedures have advantages. In the first case, the students have in front of them at all times, the ideas proposed by their classmates. In the second case, the ideas may be springing

up so quickly that to stop them to write them out would simply destroy the student's train of thought.

The hypotheses may be statements which are simply right or wrong, or they may be statements which must be qualified or elaborated upon. The latter seems to provoke more discussion and interest. For example, when Bob stated his hypothesis as "There are two answers to each problem except . . . " (see Chapter IV, p. 47) it was no longer a simple matter of proving him right or wrong. It now became a hypotheses which had to be considered and refined.

Bob could not apply his hypothesis of factoring at the time he made it (see Chapter V, p. 62) but it was nevertheless allowed to stand. Such ideas at first glance may seem irrelevant to the other students or, and this is important, to the teacher, but they must be allowed to stand. The use of hunches in discovery cannot be underestimated. It is the individual such as Bob who can lead his classmates to great achievements with his "far-out" ideas. Bob very often could not complete his ideas or refine them. But someone else could and did. This is, after all, the essence of a team approach.

10. The specific direction the class takes depends upon the students. They are responsible for proposing, defending, verifying, altering, discarding, or applying ideas. As a result, errors may go by undetected for some time (see Chapter V, p. 73), and excellent ideas may pass by without being put to good use. However, once the students realize their responsibility, and come to realize that excellent ideas are coming from each other they will become far more

in tune with each other. Johnston says:

> The teacher must convey to the students that unless they participate in an active, energetic fashion, there will be no further class and consequently no learning. (Johnston, 1968, p. 120)

In the first few lessons, the students did let errors go by (they seemed to think that it was the teacher's job to find them), and it took them a great deal of time to learn to consider ideas which came from each other, but which had not been given the teacher's stamp of approval. For example, the idea of factoring was proposed the very first day but was not really considered until late in the second day. They had no faith yet, in their ability to produce mathematics. Like all else, this the students learned gradually.

11. Closure in the strictest sense is never really achieved. The students are never made to feel that they now know all that there is to know about a topic. Rather, new concepts about old work are accepted as natural, and the students work back and forth through the exercises as new concepts and techniques are evolved. Problems which were already done are re-examined in the light of new information. The picture then is never complete - it can always be added to.

There is no doubt that this was a notable feature of these lessons. For example, while working on Exercise #3, the students discovered a new technique and went back to problems in Exercises #1 and #2 to apply it. (see Chapter V, p. 72) The teacher encouraged this practice, and even directly promoted it by having

the students go back and categorize problems from previous exercises. Thus, the students activities were not bound by the length of a period (they worked for example for two and one-half classes on one exercise), nor were they bound to a particular exercise at a particular time.

The students may have the tendency, as we all do, to discover a "neat" little technique, apply it successfully, and then consider the matter closed. This is what happened when the students discovered how to use factoring to solve an equation such as $x^2 + 3x + 2 = 0$. (See Chapter V, p. 65) It is up to the teacher then to confront them with new mathematics which will make them re-open the topic and re-examine their notions. In this case, this is exactly what was done by the teacher. (See Chapter V, p. 69)

12. A mood or atmosphere is produced by a teacher using the mathematizing mode - one of student trust and productivity, one of trust between teacher and pupils and among pupils. An examination of the comments made by the teacher revealed that he made certain types of comments repeatedly. The belief here is that comments such as the following help create this atmosphere.

Typical teacher comments were:

1) Mary, what do you think of John's suggestion?

2) Let's see if we can find an exception.

3) That sounds like a terrific idea.

4) What would you like to discuss?

5) What happens if you change this to . . . ?

6) Let's categorize these questions.

7) How do we know which answer is right?

8) What would this question have to be to make this work correct?

9) Let's find a rule.

10) How are these two questions different?

11) How are these two questions the same?

12) Could there be any other correct answers?

It is hoped that this list of characteristics of the lessons and exercises has more clearly defined the aspirations of a teacher using the mathematizing mode, and the techniques for achieving it. Where possible, reference has been made to Johnston's description of the mathematizing mode.

What are some of the important characteristics which were discussed here and which were not explicitly discussed by Johnston? One of the most outstanding techniques used in this unit by both the teacher and pupils was the method of variation in order to find a solution, or to produce a new problem. The students obviously enjoyed asking and pursuing the question "What if . . . ?" The teacher himself initiated and encouraged this technique.

Also outstanding and peculiar to this method was the teacher and students' habit of working back and forth through the exercises. No exercise was really ever considered "finished." As more light was shed on the topic, the teacher and students went back to some of the earlier exercises to fill in the gaps. This was possible largely due to the fact that the exercises contained problems which could be looked at from many different levels.

Consequently, there was something in each exercise for all of the students.

Another important technique used by the teacher to help the student discover generalizations and organize his findings, was that of having the student categorize the problems. This technique was used to summarize and to initiate the next activity.

The theme of the entire unit was presented in the first lesson. With the exception of Exercise #4, the problems in the exercises are not ordered in any particular fashion. The student himself must organize them in terms of easiest to hardest, familiar work and new work, or in any other manner he pleases.

For a complete list of teacher and pupil behaviours as seen by Johnston, the reader is encouraged to read Appendix A.

Implications of the Analysis of Student Participation in Discussion Sessions

How much of the total class time should be spent on teacher-led discussions, and how much of the time should the pupils work on their own? Is there an ideal ratio? In an earlier study, using the mathematizing mode, it was found that three-fifths of the total time the students worked on their own and with each other. (Sigurdson, 1970, p. 88) (This particular work was done with a grade VII group.) Here, only two-fifths of the time was spent in personal-inquiry sessions. (See Chapter V, p. 86) The pupils obviously need the time to organize their ideas, to evaluate them privately, and to practice certain skills. The quality of the teacher-led discussions depends largely on the amount and quality of thinking on the pupils'

part before the discussion. More research is needed to determine how much time the pupils ideally should have to work on their own.

The results showed that four of the seventeen students made slightly over one-half of the comments during the discussion sessions which took up slightly more than half the time. (See Chapter V, p. 86) At first glance this may seem to be a very negative comment on the method itself. However, it is of utmost importance to note here that this data represents tabulations taken only during distinct discussion periods which the teacher chaired. Tabulations taken during two different personal-inquiry sessions on two different days indicated that all of the students except one discussed either with each other or with the teacher. No effort was made to determine the quality of the conversation.

Is it fair then to place emphasis on the fact that during teacher-led discussions only a few pupils made the suggestions? It may be a credit to the method that the conversation was not always a pupil-teacher dialogue, and a discredit to the study that the pupil-pupil dialogues were not recorded. Even a cursory look at the video-tapes indicates that after the first two days there was a great deal of interaction among the students. This would seem to indicate that the teacher was not the only source of information, nor the only source of analysis of the pupils' work.

It should also be remembered that different students do perform different roles. Here, for example is a general description of the more obvious characteristics of certain individual students which emerged as the lessons progressed.

Bob: Constantly proposed new ideas, some of which were rather "far-out", and which did not always seem relevant to the rest of the class. He was never afraid of being "wrong."

John: Contributed a great deal to the group during the discussions. He seemed to be concerned with making different but definitely correct suggestions.

Judy: Was especially good at evaluating hypotheses and modifying them as necessary. She helped her partner a great deal.

Frank: Often picked up other people's ideas and restated them more clearly to the group. He did not seem to credit the other students with originating the ideas. An asset to the class in terms of being a good "watch dog" - Frank checked everyone else's work for mechanical errors.

Alex: Insecure, afraid to make mistakes in front of his classmates. He vacillated between pretending to know it all, and claiming he knew nothing, neither of which was true.

Penny: Was content for the greater part of the unit to state her ideas privately to a few classmates around her or to the teacher. Penny could not defend her ideas to the class as a whole until the last few days. She was, however, a definite leader to four or five girls in terms of proposing ideas, helping them evaluate and use them, and in generating enthusiasm.

Bertha: Did not volunteer any comments but was very willing and able to contribute when asked to do so by the teacher.

Lila: Simply not concerned with communicating or working with the whole class. Lila was, however, totally engrossed in her work and discussed it often with the teacher.

Betty: Very serious about her work. She openly admitted that she preferred being "told how to do it" and became very perturbed if the lesson went on when her ideas were still incomplete. Her contributions to the class were mostly in stating precisely what the difficulty was.

Cathy: Very shy. She worked well and communicated a great deal with her partner but was exceedingly hesitant about talking to the rest of the group.

Alice: Worked well with her partner and made comments when called to do so but never volunteered information. Her partner seemed to be the spokesman for both of them.

This evaluation or description of the students is obviously completely subjective. It is included here only to emphasize that we are working with certain raw ingredients, namely the students, who come to us with the very specific abilities, aptitudes, and attitudes of unique individuals. We must therefore expect that different students respond and therefore contribute in different ways.

On the other hand, Sigurdson says:

> . . . it is of vital importance that students display their cognitive structure in the form of hypotheses to be tested. The more conscious they are of their cognitive structure and the more they subject it to testing, the more likely it is to be shaped efficiently and correctly. (Sigurdson, 1970, p. 133)

In a class discussion, the teacher has little way of knowing how or what the student is thinking if the student does not speak out. The teacher requires such feedback so as to be able to provide the mathematics suitable to the pupils' needs. Unless the teacher fully realizes the problems of each student, he can hardly adapt the mathematics to suit their needs. Also, an objective of the mathematizing mode is to provide the students with the practice in presenting and defending their ideas. Therefore, though a teacher may expect that some pupils will contribute more than others in class discussion, he will encourage all of them to do so. For those who can not or will not speak out during a class discussion, the teacher can attempt to have them do so on a

private basis. The teacher will probably gain more meaningful insights into the problems of his students if he works with them on an individual or small group basis.

The students worked with partners which they had chosen at the beginning of the unit. In spite of this, the four boys seemed to work almost completely as individuals, comparing their work to that of their partners very little. On the other hand, these boys ranked in the top five in terms of making comments during teacher-led discussions. (See Chapter V, p. 86) The girls, however, seemed to work a great deal with their partners and with other groups.

The following indicates how the students were paired off. The number after each name refers to the ranking of the students in terms of the number of comments they made during the teacher-led discussions.

Bob (1) - Frank (4)

John (2) - Alex (5) - Stephanie (17)

Judy (3) - Alice (13)

Penny (6) - Lorraine (12)

Bertha (7) - Virginia (15)

Lila (8) - Harriet (10)

Lucy (9) - Betty (11)

Leona (14) - Chris (16)

An examination of this pairing-off seems to lend some support to the notion that for the girls, one of the partners acted as a spokesman.

These very tentative conclusions based in large part on subjective observations could certainly be the impetus for

for further research. Do boys always dominate discussions? Do they profit from working in pairs? If the girls work in pairs, is there a tendency for one to be the spokesman? And perhaps more important, is there always a marked difference between the reactions of the boys and those of the girls? If so, can they be capitalized upon?

The results indicate that there is no relationship between the students' past or present achievement and the amount they participated in teacher-led discussions, or for that matter between their I.Q.'s and their participation in these discussions. (See Chapter V, p. 88) This discredits the theory that only the very bright or high achievers will take active part in this method. The same holds true for the girls only.

It is interesting to note that the correlation between their rankings on the first and second exams was significant in view of the fact that the second exam was given after the students were taught in a very traditional expository way for two eighty-minute classes. One notable exception to this pattern was Bob. Bob very obviously preferred the mathematizing mode and slipped from first place on the first exam to eleventh place on the second exam after just two days of traditional teaching! Obviously there is need for more research to determine what sort of pupil profits the most from the mathematizing mode.

A Summary and Discussion of the Results of the Classification

The classroom experiences of the first four lessons described in this study (see Chapter IV, pp. 46-51) were classified according to

six activities which were simply slight modifications of Johnston's four stages. (Johnston, 1968, pp. 60-61) The results indicated that these activities did take place, more or less in the order given, and that there was a cycling of the first four or five stages before the last one took place just as Johnston predicted. (See Chapter V, p. 96)

Johnston's four stages are:

1. introduction and exploration
2. hypothesizing
3. evaluation of hypotheses
4. summary and practice

The writer claims that a more accuracte description would be as follows:

1. introduction
2. exploration
3. hypothesizing
4. evaluation of hypotheses
5. summary
6. practice

A more important difference of opinion between Johnston and the writer exists in the description of each stage. Johnston claims that the activity in each stage is predominately in the form of a discussion session or of a personal-inquiry session. This study indicates that each stage can be of either or both forms.

Why was this classification not successful for the next five lessons? To answer this, let us examine the purpose of describing the mathematizing mode in terms of four stages as proposed by Johnston, (Johnston, 1968), or six activities as proposed in Chapter V. They seem to be useful in helping the teacher set up the unit. They are activities which the teacher encourages the student to partake in, but which he does not, and indeed could not, impose on them. It makes good sense to have the students thoroughly familiarize himself with the problem before attempting to hypothesize a solution, and it seems logical that evaluation of these hypotheses should come next. That practice and summary activities come next too is logical. But, on the other hand, the human mind is very quick, and an alert student may be proposing, evaluating, and altering an idea all in much less time then it takes him to explain it. Can we, or do we even want to prevent a student from evaluating a statement as it is made? It would seem not. The intent is, rather, to have the student retain an open mind and be willing to re-evaluate his ideas as new information is uncovered.

These are the activities in which the teacher wants the students to get involved. But, since the students react differently one from the other, the teacher made no attempt to impose a certain activity on all the pupils at a certain time. Indeed the opposite seems true. The teacher went to great pains to construct the exercises in such a way so as to cater to the different levels and needs of the students. And so, what was merely a practice exercise for one student, was, for another, a catalyst for discovering many new concepts.

As the unit progressed, the teacher increasingly allowed the students to dictate their own approaches to the exercises according to their individual needs. As a result some of the students were simply practicing a technique while others were busy formulating new hypotheses, while all working on the same exercise.
At this point it was no longer possible to classify the activities of seventeen individuals into one "neat" category. Indeed, it must be admitted that the procedure of classifying the activities of the first four lessons on the basis of the teacher's remarks to be a fairly crude one. It may or may not reflect the activities of the students.

The implication then is that Johnston's four stages or the six activities described by the writer serve only as a general guide in setting up a unit, and are not meant to impose a rigid structure on the lessons.

III. Limitations of the Study

There are obviously, limitations in the writer's ability to be objective, to identify the behaviour crucial to the method, and to properly describe it.

More specifically, however, this study is lacking in more detailed descriptions of the interaction of the students during personal-inquiry sessions. Even with the help of the video-tapes, the writer found it almost impossible to report, at any length, on their discussions. Such a report could have provided enormous insight into the quality of the students' thinking, and would

have provided the reader with a sounder basis for forming an opinion as to whether or not the process objectives were being achieved. As it stands, this study describes essentially only the activities which went on for three-fifths of the time. Therefore an analysis of the number of comments made by the pupils during the discussion sessions and not during the personal-inquiry sessions may produce a distorted picture. The students most active during the teacher-led discussions may or may not be the same ones as those most active when working on their own, with their partners, or in small groups.

The conclusions based on the statistical analysis may or may not be valid. They were based, after all, on statistics obtained from seventeen students who volunteered from a group of thirty-two students from an existing Mathematics 10 class. It must be remembered that the chronic skippers had already been removed, and that there were no repeaters of the course. These conclusions then must be interpreted merely as tentative ones for which there is need of further research.

The method of classifying the activities of the students on the basis of comments made by the teacher is perhaps not accurate. Clearly, better methods are needed to enable an observer to report more accurately and in greater depth on the activities of the students.

The general characteristics of the lessons and exercises as outlined in this chapter are the result of the writer's interpretation

of the experiences described, and may therefore be lacking in objectivity.*

Obviously missing in the discussion of the classroom experiences is a summary of the results of two exams given to the students. The only reference to them are the rankings of the students in terms of the results of these exams.

It should be noted that the main objectives of this unit as outlined in Chapter IV were:

1. To have the student develop his ability to create mathematics.

2. To create an atmosphere in which the student feels free to think about mathematics and to express his mathematical ideas.

3. To have the student develop certain concepts, skills, and techniques in mathematics.

Unfortunately, the testing for acquisition of facts and techniques is rather easily done, whereas the testing of the most important aspects of the child's behaviour such as creativity, intuition, divergent thinking, flexibility, the ability and willingness to test one's hypotheses, etc., is not so easily accomplished. To report on the testing of one of the objectives and not on the testing of the others, the ones which are in fact most important to this method, would throw an undesirable slant to this study. Therefore, this study concentrated on the processes of teaching and learning, and not, to any great extent on the products.

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*The writer believes, however, that the description of the mathematizing mode would have been grossly incomplete had she limited herself completely to what can be objectively proven.

It is true, however, especially in this day of "accountability", that the products of teaching methods must be examined. We must have evidence that we are producing children who are more creative, flexible, and independent. But this testing must be done very carefully.

IV. Suggestions for Further Research

The most important aspect of any teaching method is the reaction of the pupils to it. An attempt has been made in this study to describe the general group reactions to the lessons. Much more work, however, is needed in this area. We must find out what type of student benefits from the mathematizing mode. Can the benefit derived by the student be equated to the amount he participates in teacher-led discussion? Are the students who dominate the teacher-led discussions the same as those who work most actively with their classmates during work periods? Is there a marked difference between the reaction of the boys and girls? If so, can we capitalize on it? How much do the students profit from working in pairs? What is the ideal ratio of time spent on teacher-led discussion and work periods?

The above implies a need for tests to evaluate the mathematizing mode in terms of its objectives. Are we allowing the students to create mathematics? Are they given the freedom to think? Are they learning the required mathematics? The latter is fairly easy to test, the others are not. Some work has been done on this (see Taylor-Pearce, 1969 and Tobert, 1969), but more is needed.

Finally, more descriptive studies of the mathematizing mode must be done to determine its general applicability. The method must be tried

1. using different mathematical content,
2. for different levels of students,
3. for greater or shorter periods of time,
4. with fewer or more students,
5. with groups of students selected on criteria such as I.Q., achievement, sociability, social background, etc.

As a direct result of such studies, it could be determined whether or not the general characteristics of the lessons and exercises as described in this chapter are truly the crucial aspects of the mathematizing mode.

V. Summary

As a result of this study, a unit of mathematics using the mathematizing mode was created. It has been described in terms of objectives, content, and implementation in the classroom. It has been discussed in terms of day to day teacher and pupil behaviour, in terms of the participation of the students in teacher-led discussions, and in terms of categories as suggested by Johnston. A number of generalizations have been made about the mathematizing mode and these, where possible, have been compared to Johnston's theoretical statements. As a result, some of Johnston's description has been rejected, most of it has been supported, and a few important ideas have been added to it.

VI. Conclusion

R.C. Buck includes the following as one of the aims of education:

> To convey the fact that mathematics, like everything else, is built upon intuitive understandings and agreed-upon conventions and that these are not eternally fixed . . . Mathematics is not only received by the mind but is created by it. Discussion, often a rarity in some classrooms, can illuminate by showing that there are varied viewpoints, that communication is difficult if assumptions are not acknowledged, that there are no absolutes, and that judgement must often be suspended. (Buck, 1965, pp. 5-6)

"Mathematics is not only received by the mind but is created by it." These words undoubtedly reflect the principal aim of the mathematizing mode. Hopefully this aim was clearly reflected in the descriptions of the unit. The students did create mathematics. They presented and defended their ideas, criticized and evaluated them, and learned to alter them often. This was not the situation of the student duly memorizing algorithms designed to produce the right answer at the right time. But it was a situation in which the students actively inquired and explored. If education occurs when students are stimulated to think, then one must conclude here that education was going on.

It is hoped that a philosophy as well as a technique of teaching has emerged. True, there are very certain steps which can be taken in teaching, but one must recognize that teaching is as much an art as it is a science. Given the strong belief that mathematics is a human activity, and that children are human and therefore able

to create mathematics as well as receive it, and given broad suggestions as how to apply the mathematizing mode as well as specific examples, then it is hoped that the teacher will be able to work out a unit peculiar to himself and the given situation.

The writer has attempted to describe the mathematizing mode in a classroom situation, to analyze teacher and pupil behaviour, to extract general characteristics of the method, and where possible to compare these to Johnston's findings. (Johnston, 1968) The students, however, were the ones who directly experienced the method and it is therefore their reactions which are the most important. The writer wishes to present their comments as a conclusion to this study - they perhaps reveal more about the atmosphere created during this unit than pages of "objective" data ever could. As a final word to this study then, here are some of the comments made by the pupils.

Bob: It was very different because before the teacher taught us but in this course we really had to come up with the ideas ourselves which made us think so we had to know what we were talking about.

Frank: It wasn't a boring class. I could state something and have it proven out rather than being told by the teacher it was wrong.

Penny: In the last two weeks you had to find the answers or how to do problems by yourself or with friends. It involved you more instead of a regular class where the teacher does the talking and gives you examples for factoring. You can learn more and remember it when you have to find the formulas than when given to you.

Bertha: You could talk freely to your friends about the problems.

John: Nothing (no lesson) was explained or taught before the exercises were given. We were left to discover for ourselves what methods were correct and useful in doing the problems.

Virginia: Whereas in our regular math class everything such as the methods, and the mistakes which could be done, before such exercises are given. It gets boring when almost the same thing is explained many times. Math was more of a challenge.

Betty: I felt that the professor knew so much more about Math than I did that I felt so inferior and therefore didn't learn anything. I feel more at home in my math room. However, more people were taking part and we combined all our ideas and it really made me think. Because I am generally slow in becoming to understand something in Math, the whole two weeks was not enough for me to learn anything. I was not yet too sure about doing one thing and we went on to another. I think this is a good way to teach people who do really good in Math but for someone who doesn't pick things up so fast it leads to greater confusion.

Lorraine: It brought me back to when I really enjoyed Math, even though at times I was discouraged and wanted to go back to the regular program of learning. I think I learned a lot more from this program than some years of the book learning of math.

Harriet: The class seemed united in finding the problems out together; usually in a regular program, half the class are apathetic. The anticipation of finding the correct solution to a problem thrilled me, and got me interested in math as a challenge instead of a chore.

Judy: Without using the text book, I learned quite a lot.

Leona: The persons who aren't as progressed in math, and I consider myself among them, are better off with the regular class.

Alice: It was a very open class. What I mean by open is that you can at least state your views and methods of attacking a problem without being shot down or laughed at for not following the methods you had been given years before. And if your answer was wrong at least you can see where you went wrong because you can see where the problem won't work out.

Cathy: I liked working with friends!

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APPENDIX A

TEACHER AND PUPIL BEHAVIOUR

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The following is a list of the generalizations made by Johnston concerning teacher and pupil behaviour. His references to specific examples in his study are not included. (Johnston, 1968, pp. 119 - 124)

Teacher Behaviours

1. The teacher must wait for the students to verbalize the statement of the problem.

2. If the students do not pose a significant problem, the teacher must supply the problem within a broad context to allow at least for student participation in exact formulation of the problem.

3. The teacher should accept different problems arising from the same situation as in the brainstorming session of stage two. Every suggestion must be accepted and recorded.

4. The teacher should accept wrong or partially correct hypotheses with equal enthusiasm.

5. The teacher must convey to the students that unless they participate in an active, energetic fashion, there will be no further class and consequently no learning.

6. The teacher should encourage the use of intermediate language.

7. The teacher must be willing to accept several methods of solution to the problem.

8. In addition to stating problems, the teacher will have to state solutions to problems. The latter especially refers to definitions and conventions.

9. The teacher should not hesitate to assign problems before the students know formulas for solving them.

10. The teacher should be prepared to leave an activity before it is completely resolved, and proceed with another. In addition, he should not become anxious if some students develop inexact or incomplete ideas.

Pupil Behaviours

1. In their individual work habits, the students should exhibit a "mind set" that mathematics means looking for best ways, short-cuts, patterns, or generalizations in the solution of problems.

2. The student should exhibit a willingness to contribute answers, basically thinking out loud during class discussion. Their hypotheses will not at first be well formulated, but are presented so that other students will get the benefit of their first-order thinking.

3. The student should not only be continually looking for solutions to the problems at hand, but also stating new and significant problems.